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TEST OF HOMOGENEITY BASED ON GEOMETRIC MEAN OF VARIANCES

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Abstract

Prior to comparison of means, there is k-population variances need to be tested. The usual contention is that $\sigma_i^2 = \sigma^2, \quad \text{for } i = 1, 2..., k$. *The propose methodology utilizes the Geometric Mean among sample variances to estimate the pooled variance,* σ_*^2 *that plays a vital role in the final computation in the z-statistic. When the null hypothesis is false, this statistical innovation deserves to be considered as potential methodology.*

The illustration of this methodology using empirical data sets analyzed through the use of the Bartlett's test exhibited the same decisions when analyzed by this propose methodology. This means that the innovation brought about by this method captures similar utility at a minimum computational procedure. For simulated data sets with homogenous variances, the propose methodology is prone to detect heterogeneity due to artificial differences brought by large proportion of variance to its mean. For simulated data sets under the mixed distribution, the propose methodology is more sensitive to detect heterogeneity of variances.

The propose methodology has demonstrated a significantly higher power to detect differences of variances compared to the conventional Bartlett's test based on paired t-test. This methodology can be considered as an alternative statistical tool when there is no certainty to assume the homogeneity of variances prior to analysis of variances in comparing group means.

Keywords

Bartlett's Test, Homogeneity, Power of the Test, Geometric Mean, Simulation, Mixed Distribution

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1. Rationale

Analysis of variance (ANOVA) serves as a useful tool in many fields of science and technology including medicine. The tool plays a significant rule in comparison purposes as well as determining closeness, similarity and differences. One of the rigorous assumptions of ANOVA is the equality among variances of the supposed groups to be compared. This is popularly known as the test of homogeneity of among variances. It is advisable to pursue testing homogeneity when there is reasonable doubt that its validity and soundness casts uncertainty. This is imperative considering that such an assumption plays a vital role in the final statistical decision.

Like any other statistical tool, it has some underlying assumptions need to be met. As such, failure to satisfy the given assumptions seems to invalidate the utility of the tool. Moreover, the range test together with orthogonal comparison as consequential analyses to ANOVA may shed off questionable outcomes. There is no shortcut remedy to iron-out such problem but to undergo prior scrutiny whether testable assumption has been met. Typically, this is the object required to be undertaken prior to performing analysis of variance. Subsequently, is reduces problem in further analysis.

Literature shows that Bartlett's test provides a solution regarding problem of homogeneity. Levene's (1960) proposed another methodology which is to determine the equality of variances using the mean absolute deviation as a modification of the Bartlett's test. Brown and Forsythe (1974) also proposed another methodology wherein the median absolute deviation was used to determine the significant differences among variances.

In this study, a different methodology is being proposed with the use of Geometric Mean among the sample variances instead of using their pooled estimated as in the case of Bartlett's test. This proposition is based on the prior contention that variances are independent from sample to sample. The product of their convolution of information may be taken as root, thus, the Geometric Mean plays a significant role to this effect. It is in this light that this paper intends to address the posed problem of testing homogeneity. Here, there are k-population variances assumed to be equal. The usual contention is that $\sigma_i^2 = \sigma^2, \quad \text{for } i = 1, 2..., k$. This is the underlying assumption on the equality among population variances. Such assertion is tested based on the available information contained from their sample variances $(s_1^2,\ s_2^2, \ldots, s_k^2)$.

The methodology hereby proposed known to be geometric method which is tractable and can easily be applied to a large set of data. It utilizes the property that $(n-1)s^2/\sigma^2 \approx \chi^2$ with n-1 degrees of freedom. The expression has the following parameters; expected value of (n-1) and variance of 2(n-1).

In general the concept measures the closeness or similarity (distance) of sample variances according to some defined calculates. The propose methodology is given in the following calculations.

Let the statistic $\sigma_*^2 = \sqrt[k]{s_1^2 \bullet s_2^2 \bullet ,,, \bullet s_k^2}$ be the pooled estimator of the sample variances, if the different sample variances are all equal then the σ_*^2 serves as the best estimate that is the case where $s_1^2 = s_2^2 =,,, = s_k^2$ which assures that the null hypothesis, $H_o{:}\sigma_i^2 = \sigma^2$ is true. The statistic σ_*^2 merits statistical appeal being derived as root of the product coming from independent samples. In practice, however, this scenario seldom appears to be true. In fact its assumption of equality needs to be tested or validated. This proposed methodology hinges from the theoretical derivation such as $(n-1)s^2/\sigma^2 \approx \chi^2$.

2. Objectives of the Study

In the analysis of variance, the homogeneity among variances are always assumed without due consideration of its validity and the consequence when it is false. This study is conducted to address the problem of homogeneity of variances with a new methodology:

i) To use the Geometric Mean among sample variances as statistic in the test of homogeneity;
ii) To determine the power of the test based on Geometric Mean of sample variances over the Bartlett's test.

3. Methodology

Most analysis of variance has the main contention to determine whether population means are significantly different from each other based on the available information contained on the sample data. Basic assumption is independent from sample to sample. An equally important assumption is the homogeneity of variances as well of the different samples. This is a captivating problem that this paper intends to address. The present conventional methodology employs the Bartlett's test in addressing such problem. The derivation advanced with the following preliminary computation

$$M = v(a \ln \bar{s}^2 - \sum_{i=1}^{a} \ln s_i^2). \qquad \text{Equation (1)}$$

$$C = 1 + (a+1)/3av \qquad \text{Equation (2)}$$

when the number of replications (v) are all equal. Slight adjustment or modification when the replications (v_i) are unequal with estimates

$$M = (\sum_{i=1}^{a} v_i) \ln \bar{s}^2 - \sum_{i=1}^{a} v_i \ln s_i^2 \qquad \text{Equation (3)}$$

$$C = 1 + (1/[3(a-1)])(\sum_{i=1}^{a} 1/v_i - 1/\sum_{i=1}^{a} v_i) \qquad \text{Equation (4)}$$

where $\bar{s}^2 = \sum_{i=1}^{a} v_i s_i^2 / \sum_{i=1}^{a} v_i$, here the v_i and a are the degrees of freedom and the number of groups to be compared, respectively. The test statistic approaches to a $\chi^2 = M/C$ and approximately follows a $\chi^2 -$ distribution with $a-1$ degrees of freedom. In like manner, Levene, (1960) also proposed a methodology of determine testing the equality of variances using the mean absolute deviation as a modification of the Bartletts test.

3.1 Time and Place of Study

The study was conducted at the Central Philippines State University, Kabankalan City, Negros Occidental, Philippines. Consonant to a basic research, reading to literature review has been done exhaustively in the said University from June 1, 2015 to December 31, 2015.

3.2 Research Design an Data Gathering

The research design used was simulation technique to compare the propose methodology with the conventional Bartlett's test. The comparative performance of the two methodologies to reject false null hypotheses was tested using the paired t-test. This measures their respective power of the test. Two empirical data sets were analyzed by both methodologies. Simulated data with equal variances were generated and analyzed by both methodologies. In like manner, simulated data sets that contain four equal variances imputed with a data set that contains a different variance were analyzed by both methodologies.

3.3 Statistical Treatment of Data

In the illustration of the methodology using simulation, there were two types being generated and analyzed. The first type of data sets was homogenous distributions (normal distributions). To tests the methodology, there were 12 different data sets being tried-out. In each data set, there were five groups with five replications. There were 500 runs being generated for every data set. The comparison of the methodologies was based on the results after the 500 runs of every data set. The significant difference between the two methodologies was compared with the use of the paired t-test. Results were presented in table 1.

The second type of data sets was generated based from mixed distributions. This time, there were four homogenous distribution (normal distributions) imputed with one distribution (normal distribution) having the same mean but different variance. Finally, the data set considered to be the mixed distributions constituted five groups with five replications. The approach assured that there exists one group that is different from the rest. There were 500 runs being generated for every data set. Out of 500 data sets, 100 of which were imputed with a different variance. This assured that there were 1000 data sets with different variance from the 5000 data sets in the entire mix distribution. To test the methodology, there were ten data sets being tried-out. The comparison of the methodologies was based on the results after the 500 runs of every data set. The significant difference between the two methodologies was compared using the paired t-test. Results were presented below in table B.2 below.

The critical region of the Bartlett's test was approximated using Chi-Square distribution with 4 degrees of freedom at 0.5 percent ($\alpha = 0.005$) significance level. The null hypotheses were rejected when the computed chi-Square value is greater than Chi-Square tabulated value of 14.86. In the case of geometric method, the critical region is approximated by the normal distribution with significance level of 0.5 percent ($\alpha = 0.005$) where the z-value is 2.578. Here the variance σ^2 is substituted with the statistic σ_*^2 .

Let $T_i = s_i^2 / \sigma_*^2$ Equation (5)

be the relative deviation of the ith sample variance to its true variance. Thus, $\sum_{i=1}^{k} T_i = \sum_{i=1}^{k} s_i^2 / \sigma_*^2$. In this since, $T_i = s_i^2/\sigma_*^2$ follows a Chi-Square distribution with the following properties:

$$E(T_i) = E(s_i^2 / \sigma_*^2) = E(\chi^2)/k-1 = (k-1)/(k-1) = 1 \quad \text{Equation (6)}$$

$$Var(T_i) = Var(s_i^2 / \sigma_*^2) = Var((\chi^2)/k-1) = 2(k-1)/(k-1)^2 = 2/k-1 \quad \text{Equation (7)}$$

4. Result and Discussion

The comparative performances of the methodologies were shown in the illustration of their respective procedure. Results on the analyses of empirical and simulated data sets were presented in tables below indicating their ability to reject the null hypothesis.

A. Illustration Based on Empirical Data

Table 1: *Analyses of Homogeneity Using the Bartlett's Test. Taken from Snedecor and Cochran (1980, pp 252-253)*

Sample Data Set/ Problem Set	s_1^2	s_2^2	s_3^2	s_4^2	s_5^2	**Decision**
Data Set 1 (Table 13.10.1)	178	60	98	68	-	homogeneity of variances
Data Set 2 (Table 13.10.2)	0.909	0.497	0.076	0.103	0.146	Non homogeneity of variances

Data sets presented in table 1 are empirical data sets. Testing their homogeneity, the results showed that Bartlett's test provides decision of homogeneity among variances in data set 1. This means that there is no significant difference among variances being compared. For data set 2, the Bartlett's test provides decision that there exists at least one variance that is significantly different from the rests. Using the same data sets, the propose methodology was used and the procedure was presented in tables below.

A.2 Analyses **of Homogeneity Using the Geometric Mean Among the Variances Test.** Taken from Snedecor and Cochran (1980, pp 252-253)

Table 2: *Sample Data Set 1 with the use of Geometric Mean of variances for testing homogeneity*

Sample	s_i^2	$T_i = s_i^2 / \sigma_*^2$	$T_i - 1$
1	178	1.937953902	0.937953902
2	60	0.653242888	-0.346757112
3	98	1.066963384	0.066963384
4	68	0.74034194	-0.25965806
$\sigma_*^2 =$	$(178*60*98*68)^{(1/4)}$	$\sum_{i=1}^{4}(T_i - 1) =$	0.398502114
$\sigma_*^2 =$	91.84945		
		$z_c =$	$\frac{\sqrt{\sum_{i=1}^{4}(T_i - 1)}}{\sqrt{2/(k-1)}}$
	Homogeneity of variances	$z_c =$	0.77314448^{ns}

In data set 1, the propose methodology shows that there is homogeneity among variances. This is indicated by a z-value of 0.77 which is not significant. This result speaks to mean that the variances compared are not significantly different from each other. This is consistent with the result performed using the Bartlett's test.

Table 3: *Sample Data Set 1 with the use of Geometric Mean of variances for testing homogeneity*

Sample	s_i^2	$T_i = s_i^2 / \sigma_*^2$	$T_i - 1$
1	0.909	4.130278714	3.130278714
2	0.497	2.258249198	1.258249198
3	0.076	0.345325833	-0.654674167
4	0.103	0.468007379	-0.531992621
5	0.146	0.6633891	-0.3366109
$\sigma_*^2 =$	$(0.909*0.497*0.076*0.103*0.146)^{(1/5)}$	$\sum_{i=1}^{4}(T_i - 1) =$	2.865250225
$\sigma_*^2 =$	0.220082134		
		$z_c =$	$\frac{\sqrt{\sum_{i=1}^{5}(T_i - 1)}}{\sqrt{2/(k-1)}}$
	Non homogeneity of variances	$z_c =$	2.393846277*

Data sets presented in tables A.2.1 and table A.2.2 are empirical data sets. Testing their homogeneity, the result showed that the propose methodology attained the same decisions with that of Bartlett's test. In data set 1, the propose methodology provides decision that there is no significant difference on variances. Similarly, for data set 2, the propose methodology provides decision that there exists at least one of the variances that is significantly different from the rests.

In both data sets, the propose methodology performs equally with the conventional Bartlett's test. However, it can be gleaned that the propose methodology is more tractable compared to that of the Bartlett's test.

B. Illustration of the Methodology Based on Simulated Data

Table B.1 shows the simulated data sets generated from independent identically distributed normal distribution. These data sets are all homogenous. Table B.2 shows the simulated data from normal distribution, however, imputed with a normal distribution with a different variance.

Table B.1: *Homogenous distribution with five groups each with five replications. Twelve data sets with various means and variances simulated 500 times. Equality of variances has been tested using the Bartlett's test and the methodology applying Geometric Mean of variances.*

Homogenous Distribution	Decision of Rejecting H_o	
	Bartletts Test	**Geometric Mean**
$N(\mu = 50, \sigma^2 = 4)$	0	1
$N(\mu = 50, \sigma^2 = 9)$	0	3
$N(\mu = 50, \sigma^2 = 16)$	1	4
$N(\mu = 50, \sigma^2 = 25)$	1	5
$N(\mu = 50, \sigma^2 = 36$	2	5
$N(\mu = 50, \sigma^2 = 49)$	2	6
$N(\mu = 100, \sigma^2 = 4)$	0	1
$N(\mu = 100, \sigma^2 = 9)$	0	3
$N(\mu = 100, \sigma^2 = 16)$	1	4
$N(\mu = 100, \sigma^2 = 25)$	2	5
$N(\mu = 100, \sigma^2 = 36)$	3	7
$N(\mu = 100, \sigma^2 = 49)$	3	10
t-test p-value =0.000241	15	54

Table B.1 shows the test for homogeneity among variances from five groups with five replications. At the start there was no significant difference in means and variances considering that the sets were generated from the same distributions. Homogeneity was tested using the conventional Bartletts test and the propose methodology known as the geometric method. The critical region of the Bartlett's test was approximated using Chi-Square distribution with 4 degrees of freedom at 0.5 percent ($\alpha = 0.005$) significance level. The null hypotheses were rejected when the computed chi-Square value is greater than Chi-Square tabulated value of 14.86. In the case of geometric method, the critical region is approximated by the normal distribution with significance level of 0.5 percent ($\alpha = 0.005$) where the z-value is 2.578.

Employing the two methodologies, the analyses revealed that the conventional Bartlett's test tend to reject a true null by about 15 out of 6000 or 0.25 percent. While the propose methodology tend to reject the null hypothesis by about 54 out of 6000 or 0.9 percent. Both methodology tend to detect heterogeneity from simulated data most especially when the variance is proportional to the mean. This was particularly noticeable when the variance range from ¼ to

1/3 up to ½ of the mean. This is especially true for the fact that when the variance of the simulated data gets larger there will be wider discrepancies among observations as in the case of $N(\mu = 100, \sigma^2 = 49)$. Thus, superficially creates wider gaps between variances from group to group thereby tend to produce heterogeneity. However, for short-tailed distribution where the distribution is $N(\mu = 100, \sigma^2 = 4)$ there is less chance to create a superficial heterogeneity.

Table B.2: *Mixed distributions with four groups of homogeneous distribution with five replications and an imputed distribution with five replication. Ten data sets with various means and variances simulated 500 times. Equality of variances has been tested using the Bartlett's test and the methodology applying Geometric Mean of variances.*

Homogenous Distribution	Imputed Distribution	Decision of Rejecting Ho	
		Bartlett's Test	Geometric Mean
$N(\mu = 50, \sigma^2 = 4)$	$N(\mu = 50, \sigma^2 = 36)$	40	79
$N(\mu = 50, \sigma^2 = 9)$	$N(\mu = 50, \sigma^2 = 36)$	35	70
$N(\mu = 50, \sigma^2 = 16)$	$N(\mu = 50, \sigma^2 = 36)$	12	25
$N(\mu = 50, \sigma^2 = 25)$	$N(\mu = 50, \sigma^2 = 36)$	7	11
$N(\mu = 50, \sigma^2 = 49)$	$N(\mu = 50, \sigma^2 = 36)$	16	23
$N(\mu = 100, \sigma^2 = 4)$	$N(\mu = 100, \sigma^2 = 36)$	20	76
$N(\mu = 100, \sigma^2 = 9)$	$N(\mu = 100, \sigma^2 = 36)$	41	70
$N(\mu = 100, \sigma^2 = 16)$	$N(\mu = 100, \sigma^2 = 36)$	35	68
$N(\mu = 100, \sigma^2 = 25)$	$N(\mu = 100, \sigma^2 = 36)$	39	52
$N(\mu = 100, \sigma^2 = 49)$	$N(\mu = 100, \sigma^2 = 36)$	27	43
t-test p-value = 0.001155		272	517

As shown in table B.2, there were 5000 heterogeneous data sets of various distributions. The equality of their variances were tested using the conventional Bartletts test and the propose methodology known as the geometric method. The critical region of the Bartlett's test was approximated using Chi-Square distribution with 4 degrees of freedom at 0.5 percent ($\alpha = 0.005$) significance level. The null hypotheses were rejected when the computed chi-Square value is greater than Chi-Square tabulated value of 14.86. In the case of geometric method, the critical region is approximated by the normal distribution with significance level of 0.5 percent ($\alpha = 0.005$) where the z-value is 2.578.

The null hypotheses of homogeneity showed that there were 272 out of 1000 data sets with different variance or about 27.2 percent of the false null hypotheses that were rejected based on Bartlett's test. Using similar data sets, the equality of variances were tested using the propose methodology exploiting the information from the Geometric Mean among variances. The analyses revealed that there were 517 out 1000 data sets with different variances or about 51.7 percent of the false null hypotheses that were rejected with this methodology. The null hypotheses of homogeneity of variances were rejected using z-test shown in equation (10), Appendix A.

When the variances of the mixed distributions tend to overlap each other, the two methodologies can detect only fewer data sets with problem of heterogeneity. However, when the variances of the mixed distributions do not overlap each other up to two standard deviation as in the case of $\sigma^2 = 4$ and $\sigma^2 = 36$, the two methodologies easily detects problem of heterogeneity. Further analysis of determining differences of the two methodologies, the data revealed that the propose methodology is much better in detecting the existence of heterogeneity among variances. This is indicated by the p-value of 0.001155 based on the paired t-test.

5. Summary, Conclusion and Recommendations

The methodology presented is more tractable compared to that of the Bartlett's test and the Levene's test methodology. Most alternative tests to statistics assume the distribution-free statistical methodology. As a result, this eventually reduces the power of the test. In the case of testing homogeneity of variances, this indeed assumes that the sample data are coming from normal distributions. The Geometric Mean among sample variances estimates the population variance plays a vital role in the final computation in the z-test. When the null hypothesis is false, then this methodology seems to be appealing.

The illustration of this methodology using sample data sets analyzed through the use of the Bartlett's test exhibited the same decisions when analyzed by this methodology for empirical data. In the case of simulated data sets, the propose methodology is more sensitive to reject false null hypotheses than the conventional methodology. This means that the innovation brought about by this method captures similar utility at a minimum computational procedure.

The propose methodology has manifested a significantly higher indication to detect differences of variances compared to the conventional Bartlett's test. It can therefore be deduced that this methodology be adopted when there is no certainty to assume the homogeneity of variances prior to analysis of variances in comparing group means. It is also recommended to modify the Bartlett's test using the Geometric Mean among sample variances instead of the usual mean of variances shown in equation (1) and (3).

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Appendix A Derivation of the mean and variance of the estimator

Given **k** groups to compare(from equation 5), let

$$\sum_{i=1}^{k} T_i = \sum_{i=1}^{k} s_i^2 / \sigma_*^2$$

$$E(\sum_{i=1}^{k} T_i) = E(\sum_{i=1}^{k} s_i^2 / \sigma_*^2)$$

$$= E(\frac{s_1^2}{\sigma_*^2} + \frac{s_2^2}{\sigma_*^2} + ... + \frac{s_k^2}{\sigma_*^2})$$

$$= E(\frac{s_1^2}{\sigma_*^2}) + E(\frac{s_2^2}{\sigma_*^2}) + ... + E(\frac{s_k^2}{\sigma_*^2})$$

$$= E(\frac{\chi_{k-1}^2}{k-1}) + E(\frac{\chi_{k-1}^2}{k-1}) + ... + E(\frac{\chi_{k-1}^2}{k-1})$$

$$= \frac{k-1}{k-1} + \frac{k-1}{k-1} + ... + \frac{k-1}{k-1},$$ with k times, hence

$$E(\sum_{i=1}^{k} T_i) = k \qquad \text{Equation (8)}$$

with the variance of

$$Var(\sum_{i=1}^{k} T_i) = Var(\sum_{i=1}^{k} s_i^2 / \sigma_*^2)$$

$$= Var(\frac{s_1^2}{\sigma_*^2} + \frac{s_2^2}{\sigma_*^2} + ... + \frac{s_k^2}{\sigma_*^2})$$

$$= Var(\frac{s_1^2}{\sigma_*^2}) + Var(\frac{s_2^2}{\sigma_*^2}) + ... + Var(\frac{s_k^2}{\sigma_*^2})$$

$$= Var(\frac{\chi_{k-1}^2}{k-1}) + Var(\frac{\chi_{k-1}^2}{k-1}) + ... + Var(\frac{\chi_{k-1}^2}{k-1})$$

$$= \frac{2(k-1)}{(k-1)^2} + \frac{2(k-1)}{(k-1)^2} + ... + \frac{2(k-1)}{(k-1)}$$

$$= \frac{2k(k-1)}{(k-1)^2}$$

$$Var(\sum_{i=1}^{k} T_i) = \frac{2k}{k-1} \qquad \text{Equation (9)}$$

The test statistics follows the standard normal

$$z_c = \frac{\sqrt{\sum_{i=1}^{k}(T_i - E(T_i))}}{\sqrt{(Var\sum_{i=1}^{k} T_i) / k}}$$

$$z_c = \frac{\sqrt{\sum_{i=1}^{k}(T_i - E(T_i))}}{\sqrt{\frac{2k}{k-1} / k}}$$

$$z_c = \frac{\sqrt{\sum_{i=1}^{k}(T_i - 1)}}{\sqrt{\frac{2}{k-1}}}$$