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QUENCHING BEHAVIOR OF THE SOLUTION FOR THE PROBLEMS WITH SEQUENTIAL CONCENTRATED SOURCES

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Abstract

This article studies the diffusion problems with a concentrated source which is provided at a sequential time steps in 1 dimensional space. The problems are considered for both Gaussian and fractional diffusion operators. For the fractional diffusion case, Riemann-Liouville operator with fractional order is used to describe the model with diffusion rate slower than normal time scale, which is known as sub diffusive problems. Due to this sub diffusive property, the existence and nonexistence behavior of the solution will be studied. Since the forcing term will experience a concentrated source at a sequence of time steps, the frequency, the time difference and strength of the source may affect the growth rate of the solution. Criteria for these effects which may cause for the quenching behavior of the solution will be given. The existence of the solution is investigated. The monotone behavior in spatial will be given. The quenching behavior of the solution will be studied. The location of the quenching point will be discussed.

Keywords:
Green Function, Heat Operator, Fractional Diffusion Equations, Sequential Concentrated Source, Quenching, Quenching Points

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1. Introduction

Let $\alpha \leq 1$, τ, T, a, b, c, d be positive real numbers, $L_\alpha u = u_t - (D_t^{1-\alpha} u)_{xx}$, where $D_t^{1-\alpha} u$ denotes the Riemann-Liouville derivative with fractional order, when $\alpha < 1$ and L_1 is the heat operator. Let $D_a = (0, a)$ be the finite interval on **R**. For positive integer N and $k = 1,2, \cdots, N$, let $t_k = k \cdot \tau$.

We consider the problem

$$L_\alpha u(x,t) = d \sum_{k=1}^{N} \delta(t - t_k) f\big(u(x,t)\big), \tag{1}$$

$(x,t) \in D_a \times (0,T)$, with initial condition

$$u(x,0) \equiv 0, \tag{2}$$

The Dirchlet boundary conditions are posed as

$$u(0,t) = 0 \wedge u(a,t) = 0 for t > 0. \tag{3}$$

Note that $\delta(t)$ denotes ~~is~~ the Dirac Delta function, and $f(u) > 0, f'(u) > 0, f''(u) \geq 0$, for $u \geq 0$, $\lim_{u \to c^-} f(u) = \infty$.

2. Literature Review

The condition for the forcing term $f(u)$ becomes unbounded when the solution u approaches c was introduced by Kawarada (1975) and is known as quenching behavior of the solution. These behavior for the Gaussian diffusion problems were studied by many mathematicians, and serval different directions of investigation are inspired. For example, Chan and Kong (1995) used the quenching model to describe the sudden rapid reaction rate change at explosion; Chan (2011) showed the criteria for the solution of parabolic problem to quench in multi-dimensional space. Some mathematicians studied the effect of the domain size on the quenching behavior of the solution. In particular, for n=1, the critical length is known as the length of spatial domain for which the solution exists for all time when $a < a^{\cdot}$, and quenches when for $a > a^{\cdot}$. Fruitful results in theoretical and numerical sense were obtained. For $\alpha = 1$, the theoretical proof of unique critical length and a computational method of finding it were finished by Chan and Chen (1989), Chan and Kwong (1989), and Chan (1993). For n-dimensional problem, the size of the critical domains was studied by Chan (2011), Chan and Chan and Liu (2017). Furthermore, the quenching criteria for the concentrated source was studied by Chan (2011), Chan and Tragoonsirisak (2008) etc.

In this paper, the problem (1) -(2) with α, that is either the classical Gaussian diffusive operator or the fractional diffusive operator, with concentrated sources appear at sequential time steps are investigated accordingly. For $\alpha < 1$, the operator

$$L_\alpha u(x,t) = u_t(x,t) - \big(D_t^{1-\alpha} u(x,t)\big)_{xx}$$

is used to simulate the Brownian motion with diffusion rate which is slower than normal time scale. The anomalous diffusion has been used in the modeling of many fields, for example, for those of turbulence, seepage in porous media, pollution control, etc. (c.f. Ah, Angulo and Ruiz-Medina (2005) Chan (1993), Meerschaert and Tadjeran (2004), Greenenko, Chechkin and Shulga (2004), Schula and M. Schulz (2006)). In particular, we consider the situation that the domain has microscopic pores which has a low conductivity rate in a porous media material. The fractional operator is now used to model this problem in which the medium is called subdiffusive (c.f. see Metzler and J. Klafter, 2000, Podlubny, 1999, Trujillo, 2006 etc.).

The concentrated sources appear in the form $\delta(x-b)f(u)$ in 1-dimensional case can be interpreted as an energy source stationary at a particular position, and this energy is not only given at that point but also the surrounding. The energy may be accumulated in the nearby spatial region, and hence may lead to the solution reaches a critical value so that the forcing term becomes unbounded. For these concentrated source problems in 1-dimensional space subdiffusive medium, Olmstead and Roberts (2008), Chan and Liu (2018), Liu (2019) investigated the blow-up phenomena at $x=b$, that is the forcing term appears in the form of $\delta(x-b)f\big(u(x,t)\big)$ where $\delta(x-b)$ denotes the Dirac delta function, and $f(u)$ satisfies blow-up behavior. Since the diffusive rate is slower than normal scale, the energy can be accumulated much higher, and hence unboundedness of the forcing term is more likely to occur.

Furthermore, when there is extra energy supply given into the system with a very short period of time, when compare with the whole process, we can assume that these energies appear at impulsive form. Liu and Chang (2016) studied the problems with impulsive effects on the solution at constant time steps. They showed that if $\hat{t}$ denotes the time-step for the solution u experience the impulsive effects, shorter the time-step implies more frequency the impulses take effect on the solution, and also, they gave the condition for the non-existence of the solution.

For $\alpha=1$, the green functions for the problems are given as

$$G_1(x,t-\tau,\xi)=\sum_{n=1}^{\infty}\frac{2}{a}sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{n\pi\xi}{a}\right)e^{-\left(\frac{n\pi}{a}\right)^2(t-\tau)},$$

in D_a with $G_1(x,t-\tau;\xi)=0$ for $t<\tau$.

For $\alpha<1$, the operator is used to describe the Brownian motion so that its diffusive rate is on a slower than normal time scale. And hence those problems are denoted as subdiffusive problem, or problem in a sub diffusive medium. In term of the Green's function $G_\alpha(x,t-\tau;\xi)$, by considering the integral representation form, it is capable to prove the existence of a continuous solution $u$of the sub diffusive differential equation in domain $[0,a]$, and later on Liu and Huang (2018) gave similar results in ~~to~~ infinite domain.

3. Preliminary Results

Wyss and Wyss (2001) showed that the Green's function for the operator L_α, denotes as $G_\alpha(x,t-\tau;\xi)$, related with the classical diffusive Green's function $G_1(x,t-\tau;\xi)$ as

$$G_\alpha(x,t;\xi)=\int_0^\infty g_\alpha(z)G_1(x,t^\alpha z;\xi)\,dz, \qquad (4)$$

where $g_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1-\alpha-\alpha k)} \cdot \frac{z^k}{k!}$, $z > 0$, is known as Mainardi's function. This function satisfies $g_\alpha(z) \geq 0$ for $z \geq 0$, $\int_0^\infty g_\alpha(z)dz = 1$, and $g_\alpha(z)$ tends to 0 exponentially as $z \to \infty$. Furthermore, let $\mu, \nu > 0$, the Mittag-Leffler function is defined as

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu + \mu k)},$$

which is entire for $z \in C$ where C is the complex plane (cf. Haubold, Mathai and Saxena, 2011). In particular, we get $E_{1,1}(z) = e^z$. It follows from Chan and Liu (2018) that the integrals, for $n = 1,2,3,\cdots$,

$$\int_0^\infty g_\alpha(z) e^{\frac{-n^2\pi^2}{a^2}(t-\tau)^\alpha z} dz = E_{\alpha,1}\left(\frac{-n^2\pi^2}{a^2}(t-\tau)^\alpha\right),$$

and hence

$$G_\alpha(x, t-\tau; \xi) = \frac{2}{a} \sum_{n=1}^{\infty} \sin\frac{n\pi\xi}{a} \sin\frac{n\pi x}{a} E_{\alpha,1}\left(\frac{-n^2\pi^2}{a^2}(t-\tau)^\alpha\right). \quad (5)$$

We note that $G_\alpha(x, t-\tau; \xi)$ is positive when $t > \tau$ and x, ξ are inside $(0, a)$. The solution $u$of problem (1)-(3) can now be transformed into its integral form by using these Green's functions. When $t_N < T$, we have

$$u(x,t) = d \sum_{k=1}^{N} \int_0^a G_\alpha(x, t-t_k, \xi) f\big(u(\xi, t_k)\big) d\xi \text{ for } (x,t) \in \overline{D_a} \times [0, T). \quad (6)$$

To show the existence of the solution, let us consider the case for any fixed $x \in D_a$ and perform induction on k.

For $k = 1$, let us consider the problem

$$L_\alpha v = d\delta(t - t_1) f\big(v(x,t)\big) \text{ for } (x,t) \in D_a \times (0, \tilde{t_1}), \quad (7)$$

where $t_1 < \tilde{t_1} < t_2$, and assume that v on $\overline{D}_a$.

For $i = 1,2,3,\cdots$, we define $u_i(x,t) = d \int_0^a G_\alpha(x, t-t_1, \xi) f\big(u_{i-1}(\xi, t_1)\big) d\xi$ and $u_0 = 0$, for $0 < t < \tilde{t_1}$.

By the increasing nature of $f(u)$, the sequence $\{u_i\}_{i=1}^{\infty}$ forms an increasing sequence with respect to i which is bounded above by c. Hence the limit of the sequence $\{u_i\}_{i=1}^{\infty}$, say u, satisfies the integral solution. Through a similar discussion as in the proof of Theorem 4.1 of Chan and Liu (2018) and the maximum principle (Chan and Liu (2016), Liu (2016)) that u satisfies equation (7) and is unique.

For $k = 2$, let us define

$$u_i(x,t) = d \int_0^a G_\alpha(x, t-t_1, \xi) f\big(u_{i-1}(\xi, t_1)\big) d\xi + d \int_0^a G_\alpha(x, t-t_2, \xi) f\big(u_{i-1}(\xi, t_2)\big) d\xi$$

for $i = 1,2,3,\cdots$ and $u_0 = 0$, for $0 < t < \tilde{t_2}$ where $t_2 < \tilde{t_2} < t_3$. Then the sequence $\{u_i\}_{i=1}^{\infty}$ forms an increasing sequence which is bounded above by c. A similar argument as before shows that the solution u exists and unique in $D_a \times (0. \tilde{t_2})$. In the case when $u(x,t) < c$ for $(x,t) \in D_a \times [0, \tilde{t})$ with $\tilde{t} < T$, the solution $u$exists. Hence, we have the following results.

Theorem 3.1

There exists $T > 0$ such that the solution integral solution u of (1) exists for $t \in (0, T)$. If $t_q > 0$ is the sup of the existence time of u, then $\lim_{t \to t_q^- \max_x u(x,t) = c}$.

Let $T \leq \infty$ be the largest existence time for the solution u, then next theorem follows.

Theorem 3.2

The solution $u(x, t)$ is increasing with respect to $t < T$.

Proof: Let $w = u_t$, then w satisfies $L_\alpha w = d \sum_{k=1}^{m} \delta(t - t_k) f'(u) w$ where $t_m < T$ with $m \leq N$, $w(x, 0) \geq 0$ and boundary condition $w(\cdot, t) = 0$ on boundary of D_a. Then, by applying the maximum principle to the problem, we get $w \geq 0$ on $D_a \times (0, t_m)$.

Due to the maximum principles for the diffusive problems, larger the forcing term will lead to larger the solution. Therefore, in problem (1) -(3), the factor d plays an important role on the solution u, then we proved the next theorem.

Theorem 3.3

Let $u_d(x, t)$ be the solution of the problem (1) -(3) corresponding to the parameter d and T_d be its existence time. Then for $d_1 > d_2$, we get $u_{d_1}(x, t) \geq u_{d_2}(x, t)$ on $\overline{D}_a \times \left(0, T_{d_1}\right)$.

Next, to study the location of quenching points, we obtain the following results by using the symmetric property and zero boundary condition of the solution u.

Theorem 3.4

For $t > 0$, $\max_{\overline{D}_a} u(x, t) = u\left(\frac{a}{2}, t\right)$.

Proof: Let $w(x, t) = u(x, t) - u(a - x, t)$, then $w(x, t)$ satisfies

$$L_\alpha w = d \sum_{k=1}^{N} \delta(t - t_k) f'(\zeta) w,$$

$w(x, 0) = 0$, and $w(0, t) = 0 = w(a, t)$. Combining the arguments in Theorems 2.4, 2.5 of Liu (2016) that $w(x, t) = 0$. Therefore, by Mean Value Theorem we get $u_x\left(\frac{a}{2}, t\right) = 0$, $u_x(x, t) > 0$ for $0 < x < \frac{a}{2}$, and $u_x(x, t) < 0$ for $\frac{a}{2} < x < a$. The theorem follows.

4. Main Results: Quenching Criteria for the Solution

Theorem 4.1

The solution u quenches at time T when d is large enough.

Proof: The Green's function is given as

$$G_\alpha(x, t-\tau, \xi) = \sum_{n=1}^{\infty} \frac{2}{a} sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{n\pi\xi}{a}\right) E_{\alpha,1}\left(\frac{-n^2\pi^2}{a^2}(t-\tau)^\alpha\right).$$

If there is $k > N$ such that $T < t_k = k\tau$ and u quenches at $t = T$, the result follows. Otherwise, assume that $u(x,t) < c$ for $t < N\tau$. According to the increasing behavior of u with respect to t, we get $u(x,t) > 0$ and hence $f\big(u(x,t)\big) > f(0)$ for $x \in \overline{D_a}$ and $0 < t < N\tau$. Then by use of the integral representation form of the solution u, we get

$$u\left(\frac{a}{2}, t\right) = d\sum_{k=1}^{N}\int_0^a G_\alpha\left(\frac{a}{2}, t-t_k, \xi\right) f\big(u(\xi, t_k)\big) d\xi$$

$$\geq df(0)\sum_{k=1}^{N}\int_0^a \sum_{n=1}^{\infty}\frac{2}{a} sin\left(\frac{n\pi}{2}\right) sin\left(\frac{n\pi\xi}{a}\right) E_{\alpha,1}\left(\frac{-n^2\pi^2}{a^2}(t-k\tau)^\alpha\right) d\xi$$

$$df(0)\sum_{k=1}^{N}\sum_{m=1}^{\infty}\frac{(-1)^m 4}{(2m+1)\pi} E_{\alpha,1}\left(\frac{-(2m+1)^2\pi^2}{a^2}(t-k\tau)^\alpha\right).$$

Note that $\sum_{k=1}^{N}\sum_{m=1}^{\infty}\frac{(-1)^m 4}{(2m+1)\pi} E_{\alpha,1}\left(\frac{-(2m+1)^2\pi^2}{a^2}(t-k\tau)^\alpha\right)$ is bounded above by some $M > 0$, and hence is bounded above when $0 < t \leq \left(N - \frac{1}{2}\right)\tau$. For $d > \frac{c}{f(0)M}$, there is T such that $u\left(\frac{a}{2}, T\right) > c$. This gives that the solution u reaches c in a time less than T, and hence the solution quenches in a finite time.

Consequently, from Theorem 1.4, the solution attends its maximum value at the center, that is if $u(x,t)$ quenches in a finite time, then $x = \frac{a}{2}$ is a quenching point.

Theorem 4.2

For small enough d, the solution u bounded above by c for all time.

Proof: Firstly, we show that u does not quench at time $t = k\tau$ for any $k = 1, 2, \cdots, N$. Suppose not, there is k_1 such that $u\left(\frac{a}{2}, t\right) \to c$ when $t \to k_1\tau$. There is $\eta > 0$ such that

$u\left(\frac{a}{2}, (k_1-1)\tau\right) < c - \eta$. For $\left(k_1 - \frac{1}{2}\right)\tau < t < k_1\tau$ and $x \in D_a$,

$$u(x,t) \leq u\left(\frac{a}{2}, t\right) = d\sum_{k=1}^{k_1-1}\int_0^a G_\alpha\left(\frac{a}{2}, t-t_k, \xi\right) f\big(u(\xi, t_k)\big) d\xi$$

$$\leq d(k_1-1) f\left(u\left(\frac{a}{2}, (k_1-1)\tau\right)\right)\sum_{m=1}^{\infty}\frac{(-1)^m 4}{(2m+1)\pi} E_{\alpha,1}\left(\frac{-(2m+1)^2\pi^2}{a^2}(t-(k_1-1)\tau)^\alpha\right)$$

$$\leq d(k_1-1) f\left(u\left(\frac{a}{2}, (k_1-1)\tau\right)\right)\sum_{m=1}^{\infty}\frac{(-1)^m 4}{(2m+1)\pi} E_{\alpha,1}\left(\frac{-(2m+1)^2\pi^2}{a^2}\left(\frac{1}{2}\tau\right)^\alpha\right).$$

Since $\sum_{m=1}^{\infty}\frac{(-1)^m 4}{(2m+1)\pi} E_{\alpha,1}\left(\frac{-(2m+1)^2\pi^2}{a^2}\left(\frac{1}{2}\tau\right)^\alpha\right) < M$ for some M, by taking d small enough, we get

$u(x,t) \leq c - \eta$ for $\left(k_1 - \frac{1}{2}\right)\not{t} < t < k_1\not{t}$ and $x \in D_a$. This contradicts with $u\left(\frac{a}{2}, t\right) \to c$ when $t \to k_1\not{t}$.

A similar argument obtains

$$u(x,t) \leq u\left(\frac{a}{2}, t\right) \leq dNf\left(u\left(\frac{a}{2}, N\not{t}\right)\right) \sum_{m=1}^{\infty} \frac{(-1)^m 4}{(2m+1)\pi} E_{\alpha,1}\left(\frac{-(2m+1)^2\pi^2}{a^2}\left(\frac{1}{2}\not{t}\right)^{\alpha}\right)$$

for any t, which is less than c when d is small.

Combining the above theorems with Theorem 1.3, we get that there is d such that the solution u quenches in a finite time when $d > d$, and u is bounded away from c in all time when $d < d$.

5. Main Results: Quenching Criteria for the Solution of $\alpha = 1$ on D_∞

The above results are now extended to the problem in infinite domain $D_\infty = (-\infty, \infty)$, that is, consider the problem (1)-(2) with boundary condition: $|u(x,t)| \to 0$ as $|x| \to \infty$. The integral representation form is given as

$$u(x,t) = d \sum_{k=1}^{N} \int_{-\infty}^{\infty} G_1(x, t - t_k, \xi) f\big(u(\xi, t_k)\big) d\xi.$$

Similar to the finite interval situation, we have location for the maximum value of the solution.

Theorem 5.1

For $t > 0$, $\max_{D_\infty} u(x,t) = u(0,t)$.

Proof: Let $w(x,t) = u(x,t) - u(-x,t)$, then $w(x,t)$ satisfies

$$L_\alpha w = d \sum_{k=1}^{N} \delta(t - t_k) f'(\zeta) w,$$

$w(x,0) = 0$, and $|w(x,t)| \to 0$ as $|x| \to \infty$. It follows from Theorems 2.4, 2.5 of Liu (2016) that $w(x,t) = 0$. Therefore, by Mean Value Theorem we get $u_x(0,t) = 0$, $u_x(x,t) > 0$ for $-\infty < x < 0$, and $u_x(x,t) < 0$ for $0 < x < \infty$. The theorem follows.

The quenching and non-quenching criteria for the problem are given in the following theorems.

Theorem 5.2

The solution u *quenches at time* T *for* d *is large.*

Proof: The Green's function is given as

$$G_1(x, t - \tau, \xi) = \frac{1}{\sqrt{4\pi(t - \tau)}} e^{\frac{-(x-\xi)^2}{4(t-\tau)}}.$$

If there is $k < N$ such that $T < t_k = k\tau$ and u quenches at $t = T$, the result follows. Otherwise, assume that $u(x,t) < c$ for $t < N\tau$. According to the increasing behavior of u with respect to t, we get $u(x,t) > 0$ and hence $f\big(u(x,t)\big) > f(0)$ for $x \in \overline{D}_\infty$ and $0 < t < N\tau$. Then by use of the green's function and transform the solution $u$to its corresponding integral form, we get for $t > 0$,

$$u(0,t) = d\sum_{k=1}^{N}\int_{-\infty}^{\infty} G_1(0,t-t_k,\xi)f\big(u(\xi,t_k)\big)d\xi$$

$$\geq df(0)\sum_{k=1}^{N}\int_{-\infty}^{\infty} G_1(0,t-t_k,\xi)d\xi$$

$$df(0)\sum_{k=1}^{N}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi(t-t_k)}}e^{\frac{-\xi^2}{4(t-t_k)}}d\xi$$

$$df(0)N.$$

Then for $d > \frac{c}{f(0)}$, the solution u quenches in a finite time.

As a consequence, the above theorem shows that if $u(x,t)$ quenches in a finite time, and -$x = 0$is a quenching point.

Theorem 5.3

For small enough d*, the solution* u *remains bounded above by* c *for all time.*

Proof: We claim that u does not quench at time $t = k\tau$ for any $k = 1,2,\cdots,N$.

Suppose not, there is k_1 such that $u(0,t) \to c$ when $t \to k_1\tau$. There is $\eta > 0$ such that $u(0,(k_1-1)\tau) < c - \eta$. For $\left(k_1 - \frac{1}{2}\right)\tau < t < k_1\tau$ and $x \in D_\infty$,

$$u(x,t) \leq u(0,t) = d\sum_{k=1}^{k_1-1}\int_{-\infty}^{\infty} G_1(0,t-t_k,\xi)f\big(u(\xi,t_k)\big)d\xi$$

$$\leq df\big(u(0,(k_1-1)\tau)\big)\sum_{k=1}^{k_1-1}\int_{-\infty}^{\infty} G_1(0,t-t_k,\xi)d\xi$$

$$d(k_1-1)f\big(u(0,(k_1-1)\tau)\big).$$

By taking $d < \frac{c-\eta}{(k_1-1)f\left(u(0,(k_1-1)\tau)\right)}$, we get $u(x,t) \leq c - \eta$ for $\left(k_1 - \frac{1}{2}\right)\tau < t < k_1\tau$ and $x \in D_\infty$. This contradicts with $u(0,t) \to c$ when $t \to k_1\tau$.

A similar argument obtains $u(x,t) \leq u(0,t) \leq dNf\big(u(0,N\bar{t})\big)$ for any t, which is less than c when d is small.

6. Conclusions

In this study, the sequential time source problems were studied in classical heat and fractional diffusive operators as well. We showed the increasing nature of the solution with respect to time, so that the solution of the problem might reach its critical value such that the forcing terms become unbounded in a finite time, which is also known as quenching. In particular, we showed that if the weight parameter of the source is small, the energy cannot be accumulated large enough, so that the solution exists for all time. Conversely, when the parameter is large, the energy will grow fast enough for quenching to occur. These results advance the existence properties of the concentrated source problems, and deepen the understanding of the role of fractional derivatives and the singularity of solution. It also provided another analytical tool for the study of concentrated sources' problem in sub diffusive medium. The variable sequential steps problem is worth for further discussion, especially, with increasing time intervals which can be interpreted prolonging the effects of the energy sources in the physical situation.

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REFERENCES

A. A. Greenenko, A. V. Chechkin and N. F. Shulga (2004), Anomalous diffusion and Levy flights in channelling, Physics Letters A 324, 82-85. https://doi.org/10.1016/j.physleta.2004.02.053

B. M. Schula and M. Schulz (2006), Numerical investigations of anomalous diffusion effects in glasses, Journal of Non-Crystalline Solids 352, 4884-4887. https://doi.org/10.1016/j.jnoncrysol.2006.04.027

C. Y. Chan (1993), Computation of the critical domain for quenching in an elliptic plate, Neural Parallel Sci. Comput. 1 153-162.

C. Y. Chan (2011), A quenching criterion for a multi-dimensional parabolic problem due to a concentrated nonlinear source, J. of Comp. and Appl. Math., 3724-3727. https://doi.org/10.1016/j.cam.2011.01.017

C. Y. Chan and C. S. Chen (1989), A numerical method for semilinear singular parabolic quenching problems, Quart. Appl. Math., 47, 45--57. https://doi.org/10.1090/qam/987894

C. Y. Chan and H. T. Liu (2016). A maximum principle for fractional diffusion differential equations, Quart. Appl. Math., 74, 421-427. https://doi.org/10.1090/qam/1433

C. Y. Chan and H. T. Liu (2018). Existence of solution for the problem with a concentrated source in a subdiffusive medium, Journal of Integral Equations and Applications, 30, No.1 41-65. https://doi.org/10.1216/JIE-2018-30-1-41

C. Y. Chan and M. K. Kwong (1989), Existence results of steady-states of semilinear reaction-diffusion equations and their applications, J. Differential Equations, 77, 304-321. https://doi.org/10.1016/0022-0396(89)90146-0

C. Y. Chan and P. C. Kong (1995), A thermal explosion model, Appl. Math. Comput., 71, 201-210. https://doi.org/10.1016/0096-3003(94)00154-V

C. Y. Chan and P. Tragoonsirisak (2008), A multi-dimensional quenching problem due to a concentrated nonlinear source in R^N, Nonlinear Anal., 69, 1494-1514. https://doi.org/10.1016/j.na.2007.07.001

H. J. Haubold, A. M. Mathai and R. K. Saxena (2011), Mittag-Leffler functions and their applications, J. Appl. Math., Art. ID 298628, 1-51. https://doi.org/10.1155/2011/298628

H. Kawarada (1975), On solution of initial-boundary problem for $u_t=u_{xx} +1/(1-u) $, Publ. Res. Inst. Math. Sci., Kyoto Univ., 10, 729-736. https://doi.org/10.2977/prims/1195191889

H. T. Liu (2016), Strong maximum principle for fractional diffusion differential equation, Dynam. Systems and Appl., 26, 365-376.

H. T. Liu and Chien-Wei Chang (2016), Impulsive Effects on the Existence of Solution for a Fractional Diffusion Equation, Dynamic Systems and Applications, 25, 493-500.

H. T. Liu and Wei-Cheng Huang (2018), Existence of Solution for the Problem with a Concentrated Source in a Subdiffusive Medium, 6th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2017), AIP Conference Proceedings, 1926:020027.

H.T. Liu (2019), Blow-up behavior of the Solution for the problem in a subdiffusive mediums, Math. Meth. Appl. Sci., 42, No.16, 5383-5389. https://doi.org/10.1002/mma.5393

I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

J. Trujillo, Fractional models: Sub and super-diffusive, and undifferentiable solutions, in Innovation in Engineering Computational Technology, Sax-Coburg Publ., Striling, Scotland, 2006.

M. M. Meerschaert and C. Tadjeran (2004), Finite difference approximations of fractional advection-dispersion flow equations, Journal of Comp. Appl. Math., 172 65-77. https://doi.org/10.1016/j.cam.2004.01.033

M. M. Wyss and W. Wyss (2001), Evolution, its fractional extension and generalization, Fract. Calc. Appl. Anal., 4, 273-284.

R. Metzler and J. Klafter (2000), The random walk's guide to anomalous diffusion, A fractional dynamics approach, Phys. Rep., 339, 1-77. https://doi.org/10.1016/S0370-1573(00)00070-3

R. Metzler and J. Klafter (2004), The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A, 37, 161-208. https://doi.org/10.1088/0305-4470/37/31/R01

V. V. Ah, J. M. Angulo and M. D. Ruiz-Medina (2005), Diffusion on multifactals, Nonlinear Analysis 63, e2043-e2056. https://doi.org/10.1016/j.na.2005.02.107

W. E. Olmstead and C. A. Roberts (2008), Thermal blow-up in a subdiffusive medium, SIAM J. Appl. Math., 69, 514-523. https://doi.org/10.1137/080714075

W. Y. Chan (2017), Determining the critical domain of quenching problems for coupled nonlinear parabolic differential equations, Proceedings of Dynamic Systems and Applications.

W. Y. Chan and H. T. Liu (2017), Finding the Critical Domain of Multi-Dimensional Quenching Problems, Neural, Parallel, and Scientific Computations, 25, 19-28.