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SOME PROPERTIES FOR CERTAIN SUBCLASSES OF ANALYTICFUNCTIONS INVOLVING DERIVATIVE OPERATOR

Ebtisam Ali Eljamal
Al-Mergeb University, Faculty of Science, Department of Mathematics Al-Khoms, Libya
N_ebtisam@yahoo.com

MaslinaDarus
University Kebangsaan Malaysia, School of Mathematical Sciences, Faculty of Science and Technology, Bangi, Malaysia
maslina@ukm.my

Abstract

In this paper, we introduce a subclass of analytic functions by using the subordination concept between this function and generalized derivative operator. Some interesting properties of this class are obtained.

Keywords

Analytic Functions, Derivative Operator, Subordination

1. Introduction

Let A denote the class of functions of form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

Which are analytic and in the open unit disk $U = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}$. A function $f \in A$ is said to be in the class $S^*(\alpha)$ starlike functions of order α in U if and only if

$$\{\frac{zf'(z)}{f(z)}\} > \alpha \qquad (0 \leq \alpha <). \tag{2}$$

A function $f \in A$ is said to be in the class $C(\alpha)$ convex functions of order α in U if and only if

$$\{1 + \frac{zf''(z)}{f(z)}\} > \alpha \qquad (0 \leq \alpha < 1). \tag{3}$$

Let $[a, n]$ be the class of analytic functions of the form:

$$(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots (z \in U).$$

Let f,$\in A$, where $f(z)$ is given by (1) and $g(z)$ is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) $f * g$ of the functions $f(z)$ and (z) is defined by:

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z).$$

We consider the following differential operator.

***Definition 1.1*:** (see [6]). Let the function f be in the class A .For $m, \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda 2 \geq \lambda 1 \geq 0$,

we define the following differential operator

$$D^{m,\alpha}_{\lambda_{1,2}} f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right]^m C(\alpha, k) a_k z^k \tag{4}$$

It is easily verified from (2), that

$$(1 + \lambda_2(k-1))D^{m+1}(\lambda_1, \lambda_2, \alpha)(z) = (1 + \lambda_2(k-1) - \lambda_1)D^m(\lambda_1, \lambda_2, \alpha)f(z) + z\lambda_1(D^m(\lambda_{1,2,}\alpha)f(z))'. \tag{5}$$

It should be remarked that the class of differential operator $(\lambda_1,\ \lambda_2,\ \alpha)$is a generalization of several other linear operators considered in the earlier investigations (see[1]-[5]).

Let $f,\ g$ be analytic functions in U. We say that f is subordinate to g, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$, $(z \in U)$, and symbolically written as the following:

$$f \prec g(z \in U)\ or\ f(z) \prec g(z)(z \in U).$$

It is known that $f(z) \prec g(z)\ (z \in U) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$. further, if the function g is univalent in$U$, then we have the following equivalent

$$f(z) \prec (z)\ (z \in U) \Leftrightarrow f(0) = (0)$$

And $f(U) \prec (U)$.

By making use of the linear operator $(\lambda_1,\ \lambda_2,\ \alpha)$and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class$A$.

Definition 1.2: A function $f(z) \in$ is said to be in the class$\psi^{\gamma,}(m, \lambda_1, \lambda_2, \alpha, n, A, B)$ if it satisfies the following subordination condition

$$(1-\gamma)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^{\beta} + \gamma\left(\frac{D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)}{D^m(\lambda_1,\lambda_2,\alpha)f(z)}\right)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^{\beta} \prec \frac{1+Az}{1+Bz}\ (z \in U), \quad (6)$$

Where the parameters $\gamma, \beta, \alpha, \lambda_1, \lambda_2, m, A$ and B are considered as follows:

$\gamma \in \mathbb{C}, Re(\beta) > 0, \lambda_1, \lambda_2 \geq 0, \lambda_1, \lambda_2, \in \mathbb{R}, m \geq 0, -1 \leq B \leq 1, A \neq B \in \mathbb{R}$ and $n \in \mathbb{N}$. We write $\psi^{1,}(1,1,1,1, 1, -1) = \psi(\beta)$. Clearly, the class (β) is a subclass of the familiar class of Bazilevic functions of type . If we set $= 0;\ \lambda_1, \lambda_2 = 1$ in the class $\psi^{\gamma,}(m,\ \lambda_1,\ \lambda_2,\ \alpha,\ n,\ A,\ B)$, then we obtained the class studied by Liu [7]. In the resent years many researchers have studied various interesting properties with the liner operators, for example [11] and [12].

In the present paper, we aim at proving some interesting properties of the class $\psi^{\gamma,\beta}$ $(m, \lambda_1, \lambda_2, \alpha, n, A, B)$.

2. Preliminary Results

In order to establish our main results, we need the following lemmas.

Lemma 2.1: (see [8]). Let the function h be analytic and univalent (convex) in U with $h(0) = 1$. Suppose also that the function k given by

$(z) = 1 + c_n z^n + c_{n+1} z^{+1n} + \ldots$

is analytic in U. If

$$(z) + \frac{zk'(z)}{\zeta} \prec h(z)(Re(\zeta) > 0;\ \neq 0;\ z\ U), \quad (7)$$

Then

$$k(z) \prec \chi(z) = \frac{\zeta}{n} z^{\frac{-\zeta}{n}-1} \int_0^z t^{\frac{-\zeta}{n}-1} h(t)dt \prec h(t) \qquad (z \in U),$$

And (z) is the best dominant.

Lemma 2.2: (see [10]). Let (z) be a convex univalent function in U and let $\sigma, \eta \in \mathbb{C}$ with $Re\,\{1 + \frac{zq''(z)}{q(z)}\} > max\,\{0, -Re\,(\frac{\sigma}{\eta})\}$.

If the function p is analytic in U and

$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z)$,

then

$(z) \prec (z)$ and (z) is the best dominant.

Lemma 2.3: (see [9]). Let q be convex univalent in U and $k \in \mathbb{C}$. Further assume that $(k) > 0$, if

$$(z) \in [q(0),1] \cap Q,$$

and $p(z) + kzp'(z)$ is univalent in U, then

$$q(z) + kzq'(z) \prec p(z) + kzp'(z)$$

Implies $(z) \prec (z)$ and (z) is the best subdominant.

3. Main Result

Theorem 3.1: Let $f\ (z) \in \psi^{\gamma,}(m, \lambda_1, \lambda_2, \alpha, n, A, B)$ with $(\alpha) > 0$. Then

$$\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta \prec \frac{(1+\lambda_2(k-1))\beta}{\lambda_1 n\gamma}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{(1+\lambda_2(k-1))\beta}{\lambda_1 n\gamma}-1} du \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{8}$$

Proof: Define the function

$$p(z) = \left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta \quad (z \in U). \tag{9}$$

Then (z) is analytic in U with $(0) = 1$. By taking the derivative in the both sides in equality (9) and using (3), we get

$$(1-\gamma)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta + \gamma\left(\frac{D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)}{D^m(\lambda_1,\lambda_2,\alpha)f(z)}\right)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta = p(z) + \frac{\lambda_1\gamma z p'(z)}{\beta(1+\lambda_2(k-1))} \prec \frac{1+Az}{1+Bz} \quad (z \in U) \tag{10}$$

By applying Lemma 2.1 in the last equation, we get

$$\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta \prec \frac{(1+\lambda_2(k-1))\beta}{\lambda_1 n\gamma} z^{-\frac{(1+\lambda_2(k-1))\beta}{\lambda_1 n\gamma}} \int_0^z t^{\frac{(1+\lambda_2(k-1))\beta}{\lambda_1 n\gamma}-1} \frac{1+At}{1+Bt} dt = \frac{\zeta}{n}\int_0^1 u^{\frac{\zeta}{n}-1} \frac{1+Azu}{1+Bzu} du \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{11}$$

Where $\zeta = \frac{(1+\lambda_2(k-1))\beta}{\lambda_1 n\gamma}$.

The proof of Theorem 3.1 is complete.

Theorem 3.2: Let (z) be univalent in U, $0 \neq \alpha \in \mathbb{C}$. Suppose also that (z) satisfies

$$Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\frac{(1+\lambda_2(k-1))\beta}{\lambda_1\gamma}\right)\right\}. \quad (12)$$

If $(z) \in A$ satisfies the following subordination

$$(1-y)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta + \gamma\left(\frac{D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)}{D^m(\lambda_1,\lambda_2,\alpha)f(z)}\right)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta \prec q(z) + \frac{\gamma\lambda_1 zq'(z)}{(1+\lambda_2(k-1))\beta},(13)$$

Then

$$\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta \prec q(z),$$

And q(z) is the best dominant.

Proof: Let the function (z) be defined by (9). We know that (10) holds true. Combining (10) and (13), we find that

$$p(z) + \frac{\gamma\lambda_1 zq'(z)}{(1+\lambda_2(k-1))\beta} \prec q(z) + \frac{\gamma\lambda_1 zq'(z)}{(1+\lambda_2(k-1))\beta}. \quad (14)$$

By using Lemma 2.2 and (14), we get the assertion of Theorem 3.2.

Taking $(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.2, we get the following result.

Corollary 3.1: Let $\gamma \in \mathbb{C}$ and $-1 \leq B < A \leq 1$. Suppose also that $\frac{1+Az}{1+Bz}$ satisfies the condition (12). If $(z) \in A$ satisfies the following subordination

$$(1-\gamma)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta+\gamma\left(\frac{D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)}{D^m(\lambda_1,\lambda_2,\alpha)f(z)}\right)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta \prec$$

$$\frac{1+Az}{1+Bz}+\frac{\lambda_1\gamma(A-B)z}{(1++\lambda_2(k-1))\beta(1+Bz)^2}$$

Then

$$\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta \prec \frac{1+Az}{1+Bz} \text{ and } \prec \frac{1+Az}{1+Bz} \text{ is the best dominant.}$$

Theorem 3.3: Let (z) be convex univalent in U, $\gamma\in\mathbb{C}$ with $(\gamma)>0$. Also let

$$\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta \in H[q(0),1]\cap Q \text{ and}$$

$$(1-\gamma)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta+\gamma\left(\frac{D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)}{D^m(\lambda_1,\lambda_2,\alpha)f(z)}\right)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta$$

z

be univalent in U. If

$$q(z)+\frac{\gamma\lambda_1 zq'(z)}{(1+\lambda_2(k-1))\beta}\prec(1-\gamma)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta+\gamma\left(\frac{D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)}{D^m(\lambda_1,\lambda_2,\alpha)f(z)}\right)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta,$$

Then $q(z)\prec\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta$, and $q(z)$ is the best subdominant.

Proof: Let the function $P(z)$ be defined by (9). Then

$$q(z)+\frac{\gamma\lambda_1 zq'(z)}{(1+\lambda_2(k-1))\beta}\prec(1-\gamma)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta+\gamma\left(\frac{D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)}{D^m(\lambda_1,\lambda_2,\alpha)f(z)}\right)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta$$

$$=p(z)+\frac{\gamma\lambda_1 zp'(z)}{(1+\lambda_2(k-1))\beta}.$$

An application of Lemma 2.3 yields the assertion of Theorem 3.3.

Corollary 3.2: Let $q(z)$ be convex univalent in U and $-1 \le B < A \le 1$, $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$. Also let

$\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta \in H[q(0),1] \cap Q$ and

$$(1-\gamma)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta + \gamma\left(\frac{D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)}{D^m(\lambda_1,\lambda_2,\alpha)f(z)}\right)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta$$

be univalent in U. If

$$\frac{1+Az}{1+Bz} + \frac{\lambda_1\gamma(A-B)z}{(1++\lambda_2(k-1))\beta(1+Bz)^2} \prec (1-\gamma)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta + \gamma\left(\frac{D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)}{D^m(\lambda_1,\lambda_2,\alpha)f(z)}\right)\left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta,$$

Then $\frac{1+Az}{1+Bz} \prec \left(\frac{D^m(\lambda_1,\lambda_2,\alpha)f(z)}{z}\right)^\beta$, and $\frac{1+Az}{1+Bz}$ is the best subdominant.

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