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ON THE ASSESSMENT OF MATHEMATICAL COMPETENCIES FOR SOLVING NON-LINEAR EQUATIONS

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Abstract

The process of learning and acquiring mathematics by students is inextricably linked with the need to measure the achieved level of their knowledge and skills at each stage of education. The aim of the study is to propose a methodological approach for evaluating students' mathematics competencies in engineering degrees, developed in accordance with the results of the European Erasmus+ project Rules_Math. A competency assessment model is presented that includes three main components: multiple-choice test, tasks to solve and a small practical project on a topic from the curriculum for first-year students. The multiple-choice test aims to assess students' understanding of the basic notions, concepts, and methods in the field. Solving problems should measure the level of knowledge and skills acquired. The practical project could evaluate the *students' ability to apply what has been learned to solve practice-oriented tasks. It is recommended to use mathematical software. The approach is demonstrated with a specific example of assessing mathematical competencies for solving non-linear equations of a single variable. Answers and solutions are also provided. The presented methodology can be used by professors and tutors in higher education institutions to update the assessment of students' competencies in mathematics.*

Keywords

Mathematical Competency, Assessment Model, Non-Linear Equation, Learning Outcome

1. Introduction

Mathematics is at the heart of technical progress, computer applications and information technology. It holds an important place in education systems at all levels, including in higher education, for a large number of specialties - engineering, economics, computer science and more. (Aigner & Behrends, 2010). The learning of mathematics is closely linked with the assessment of the students' acquired knowledge and skills and building the ability to apply the learned in various practical situations. This is the basis of the notion of competency, as found in the documents and activities of the Program for International Student Assessment (PISA), to measure 15-year-olds' ability to use their mathematics and other knowledge and skills to meet real-life challenges (PISA, 2019). The literature abounds with publications on the assessment of student performance and the formation of mathematical competencies. Large scale assessments of PISA are studied for measuring mathematics competency at international and national level in (Ehmke, van den Ham, Sälzer, Heine & Prenzel, 2020). In (Reiss & Obersteiner, 2019) recent theoretical study is presented on how competency models can be seen as useful tools for defining, understanding, and diagnosing mathematical competency. Authors of (Mirabueno & Boyon, 2019) compare the results from different tests, measured students' mathematical ability, logical and reasoning performance in general mathematics, statistics and probability. The influence of different teaching methods on student achievement in mathematics is another important topic. For example, in a recent paper (Tokac, Novak, & Thompson, 2019) it was found through meta-analysis that video games are a slightly effective instructional strategy for teaching mathematics in secondary school. A flipped classroom approach and assessment for evaluating mathematics competencies are reported in (Llona & Tonga, 2020).

In particular, for university education, contributions to the formulation of the concept of mathematical competency developed by the Danish KOM project should be noted (Niss, 2003). In 2013, a framework for mathematics curriculum in engineering education was developed (Alpers et al., 2013). Recent publications bring new elements and summaries in this area (Niss & Højgaard, 2019).

The present study reports some recent results elaborated in the frame of the European Erasmus+ project Rules_Math as a part of the Output 2 (O2): Assessment standards and activities to implement an innovative pedagogical approach in engineering degrees (Rules_Math, 2019; Queiruga-Dios et al., 2018). The selected subject is noted as AC9: "Solution of non-linear equations", addressed to first year undergraduate students in engineering higher education (Alpers et al., 2013). The developed methodological model is designed for professors, lectures or tutors in mathematics to encourage them in introducing and incorporating the competencies-based assessment into their educational process. The assessment model includes three basic parts: multiple-choice test, questions or tasks to solve and a small practical project. The multiple-choice test comprises closed-ended questions and answers so to assess to what extent the student understands the concepts, general methods, and process of solving an algebraic equation of a single variable. This also includes the general case of nonlinear equations (polynomial and transcendental). Other elements of the test are questions about the accuracy of the calculations, as well as questions about the convergence of the basic iterative methods for solving nonlinear equations. Some questions aim to test the extent to which the student is able to apply his knowledge for solving specific equations and interpreting the result. The model also includes a small practical project that could be solved with the help of software. All elements have been developed in accordance with their compliance with the eight competencies and learning outcomes (LO) described in (Niss, 2003; Alpers et al., 2013). Some recent applications for using mathematical software to measure competencies are reported in (Kulina et al., 2018).

2. Notion of Mathematical Competency

We use the notion of mathematical competency, detailed in (Niss, 2003) for different educational levels. The current Rules_Math project aims to develop unified standards for assessing mathematical competencies for Level 1 (first-year students). The eight general mathematical competencies that our students must acquire during the first year of study are as follows.

- C1. Thinking mathematically.
- C2. Reasoning mathematically.
- C3. Posing and solving mathematical problems.
- C4. Modelling mathematically.
- C5. Representing mathematical entities.
- C6. Handling mathematical symbols and formalism.
- C7. Communicating in, with, and about mathematics.
- C8. Making use of aids and tools.

These competencies are partly overlapping and transversal at different mathematical topics and can be assessed at different levels and different types of assessment forms.

3. Objectives

The learning outcomes to be achieved by the students are detailed in Table 1. The related objectives could be summarized as follows.

1. To recognize the type of equation and the number of its roots.
2. To understand and apply different methods for locating real roots.
3. To understand the concept of reducing the localization interval and the difference with iterative processes.
4. To present the equation in a form suitable for an iterative process.
5. Apply basic methods for solving a nonlinear equation (bisection method, Newton's method and more).
6. To understand when to stop correctly the calculations of the iterative process using different convergence criteria, depending on a given accuracy level.
7. To apply the knowledge for the complete solution of practical problems for nonlinear equations.
8. To interpret the obtained result.

Table 1 shows the list of the competencies that we measure with the proposed assessment model, where green is very important, orange is medium important, and red is less important.

4. Test to Evaluate Mathematical Competencies for Solving Non-linear Equations

The time to solve this test could be 2 hours. The exam includes: multiple-choice test of six closed questions; two tasks to solve; small practical project by implementing mathematical software. The learning outcomes (LO) to be achieved with this procedure are shown in Table 1.

Table 1: *Learning Outcomes with Degree of Coverage of Competencies Involved in this Assessment Model*

Analysis and Calculus		**Competencies**							
AC9	***Solution of non-linear equations***	***C1***	***C2***	***C3***	***C4***	***C5***	***C6***	***C7***	***C8***
AC9/1	Use intersecting graphs to help locate approximately the roots of non-linear equations								
AC9/2	Use Descartes' rules of signs for polynomial equations								
AC9/3	Understand the distinction between point estimation and interval reduction methods								
AC9/4	Use a point estimation method and an interval reduction method to solve a practical problem								
AC9/5	Understand the various convergence criteria								
AC9/6	Use appropriate software to solve non-linear equations	x	x	x	x	x	x	x	x

4.1 Multiple-Choice Test

1. Which of the following is not a nonlinear equation?
   i) $3x-2=4x$.
   ii) $\sqrt{3x^5-2x}=x+1$.
   iii) $\sin(x-\frac{\pi}{3})+\cos(2x+1)-0.5=0$
   iv) All are non-linear.
2. Given the polynomial function $f(x)=2x^3-3x+4$, how many real zeros has the equation $f(x)=0$ in the interval $(-\infty,0]$?

i) 0 zeros.

ii) 1 real zero.

iii) 2 real zeros.

iv) 3 real zeros.

3. How many positive zeros could have the equation $f(x) = x^4 - x^2 + x - 2 = 0$?

i) 0 zeros.

ii) 1 or 3 zeros.

iii) 0 or 2 zeros.

iv) 4 zeros.

4. The principle of an iteration method consists in:

i) Establishing an exact formula to calculate the root of the equation.

ii) Proving the existence of a root, locating the interval, plotting the function from which you can fix the approximate value of the searched root.

iii) Using a method for representation of the equation in the so called normal form.

iv) Constructing a repeated process until an answer is achieved.

5. The difference between the interval reduction method and point estimation (fixed point iteration) approaches is:

i) The interval reduction method constructs a sequence of decreasing intervals containing the root of the equation, while the point estimation method computes successive terms using a rule formula and a starting value of the root.

ii) The interval reduction method is much faster than the point estimation method.

iii) The interval reduction method finds the exact solution, while the point estimation calculates a solution with a preset accuracy.

iv) The interval reduction method does not differ essentially in speed and accuracy to the fixed point iteration.

6. Which of the convergence criteria for calculating the real root x^* with absolute accuracy ε is applied correctly for the iteration process $x_{k+1} = g(x_k),\ k = 0,1,2,...$, where x_0 is given?

i) $|x_k - x^*| < \varepsilon$

ii) $|x_k - x_0| < \varepsilon$

iii) $|x_{k+1} - x_k| < \varepsilon$

iv) $|g(x_k)| < \varepsilon$

4.2 Question 1

Using the Newton's method find the first four iterations x_1, x_2, x_3, x_4 with $x_0 = 2$ for calculating the real root of the equation $2x^3 - 4x - 3 = 0$. Evaluate the achieved accuracy at each iteration.

Remark. When appropriate, solve this question using mathematical software.

4.3 Question 2

Using intersecting graphs locate approximately the unit root of the non-linear equation $x^3 - x + 5 = \sin(2x)$ in an interval with length 1.

Remark. When appropriate, solve this question using mathematical software.

4.4 Small Practical Project

In a chemical reaction $CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$, the percent of decomposition x per 1 mole of carbon dioxide (CO_2) is found from the equation $f(x) = \left(\frac{p}{k^2} - 1\right)x^3 + 3x - 2 = 0$, where p (atm) is the gas pressure and k is a temperature-dependent equilibrium constant. Find x for $k = 1.648$ (corresponds to a temperature of 2800 Kelvin) and $p = 1$ *atm*. Solve the equation by bisection method using mathematical software with 5 iterations.

5. Solutions and Answers

5.1 Answers to Multiple Choice Test

Solution

1. An equation is said to be nonlinear when it involves terms of degree higher than one in the unknown quantity or contains transcendental functions, such as sin, cos, log, exp, etc. The correct answer is i).
2. The function is arranged in descending powers of x. Applying the Descartes' rule of sign for negative x we have expression $f(-x) = -2x^3 + 3x + 4$ with only one change in the sign of the coefficients. So, the equation $f(x) = 0$ has one negative real zero. As $f(0) = 4 \neq 0$ then $x = 0$ is not a root. Hence the equation has exactly one real root in $(-\infty, 0]$. The correct answer is ii).

3. For positive x we have three changes in the sign of the coefficients in the given function $f(x) = x^4 - x^2 + x - 2 = 0$. The Descartes´ rule of sign tells us that we have three real positive zeros or less but an odd number of zeros. Consequently, the number of positive zeros must be either three, or one. The correct answer is ii).
4. In the first three answers there is no element of an iterative process, which consists in constructing a convergent sequence of approximate values to the solution sought. The correct answer is iv).
5. Both methods are iterative and find an approximate rather than an exact solution. This eliminates answer iii). Interval reduction method is considered to be the slowest by using the corresponding convergence theorem. Hence ii) and iv) are dropped. The correct answer is i).
6. The first answer requires knowledge of the solution sought. The second answer cannot give a convergence to 0, since x_0 is generally far from the root. The last answer is meaningless. The third answer can guarantee the convergence of the iteration process as a Cauchy sequence. The correct answer is iii).

5.2 Questions 1

Solution 1

The equation $f(x) = 2x^3 - 4x - 3 = 0$ has only one real root. Through checks we find the interval of localization $[a,b] = [1,2]$. Really, $f(a)f(b) = (-5).5 = -25 < 0$. The derivative is $f'(x) = 6x^2 - 4$. The formula of Newton's method for the given initial approximation has the form:

$$x_0 = 2,\ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},\ k = 0,1,2,3... \qquad (1)$$

We will work with intermediate accuracy $\varepsilon = 0.000001$ as we expect a fast convergence of the method. We consistently obtain the results shown in Table 2.

Table 2: *Numerical Results of the Iteration Process for Solving Question 1*

k	x_k	$\lvert x_{k+1} - x_k \rvert$	$f(x_k)$
0	2		5
1	1.75	|0.25|	0.71875

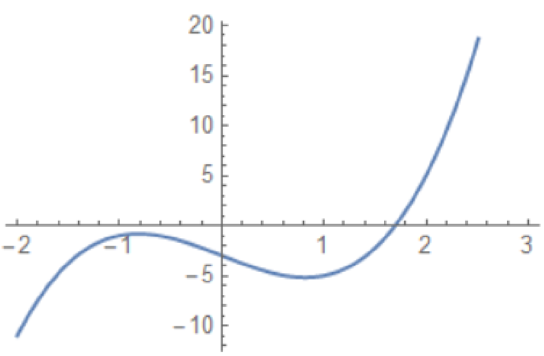
2	1.7	|0.5|	0.026
3	1.698051	|0.001949|	0.000039
4	1.698048	|0.000003|	$8.64x10^{-11}$

Answer: The approximate value of the root is $x_4 \approx 1.69805$.

Solution 2

A simple code and results in Wolfram Mathematica environment are shown in Figure 1.

```
f[x_] := 2 x^3 - 4 x - 3.   (* Define the function *)
Plot[f[x], {x, -2, 3}] (* Plot the function in the interval [-2,3] *)
fp[x_] := 6 x^2 - 4        (* Define the first derivative of the function *)
x0 = 2.;                   (* Set the initial guess *)
For[k = 1, k ≤ 4, k++,
  x1 = x0 - f[x0]/fp[x0];     (* Newton's iterative formula *)
  Print["k=", k, ", x1=", x1, ", Abs(x1-x0)=", Abs[x1 - x0],
  ", f=", f[x1]];
  x0 = x1                  (* Stores the current value in cell x0 *)
]
x1
```




```
k=1, x1=1.75, Abs(x1-x0)=0.25, f=0.71875
k=2, x1=1.7, Abs(x1-x0)=0.05, f=0.026
k=3, x1=1.69805, Abs(x1-x0)=0.00194903, f=0.0000387319
k=4, x1=1.69805, Abs(x1-x0)=2.91212×10^-6, f=8.63993×10^-11

1.698048062394615`
```

Figure 1: *Solution 2 of Questions 1 using Wolfram Mathematica*

Answer: The approximate solution is the same as it was found in Solution 1.

5.3 Questions 2

Solution

To locate a real root with the help of plots, we draw approximate plots of the two functions on the left and right sides of the equation, which we will denote by $f_1(x) = x^3 - x + 5$ and $f_2(x) = \sin(2x)$ respectively. The intersections of the two plots will show the location of the root(s).

a) Solution without computer.

The left function $f_1(x)$ has one real root. In minus infinity it tends to minus infinity, and for $x = 0$ it is positive. Therefore, the root is negative. We study the first derivative $f_1'(x) = 3x^2 - 1$. It has two roots, i.e. the function $f_1(x)$ has two local extrema, respectively, in points $x_1 = -\frac{1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}$. The local maximum is at the point x_1 and is approximately equal to 5.4. The local minimum is at the point x_2 and is approximately equal to 4.6. Also $f_1(-2) = -1$ and $f_1(-1) = 5$ so the function $f_1(x)$ grows in the interval $[-2, -1]$ and is higher than 4 in the interval The second function $f_2(x)$ varies with a period $\omega = \pi$ taking values from -1 to +1. Therefore, the root of the given equation is in the interval $[-2, -1]$.

b) Solution using software.

This is comparatively much easier with the formulas and plots parameters being set correctly. Our code and localization result is shown in Figure 2.

```
f1[x_] := x^3 - x + 5          (* Define the left side of the function *)
f2[x_] := Sin[2 x]             (* Define the right side of the function *)
Plot[{f1[x], f2[x]}, {x, -π, π}]        (* Plot the two functions *)
```

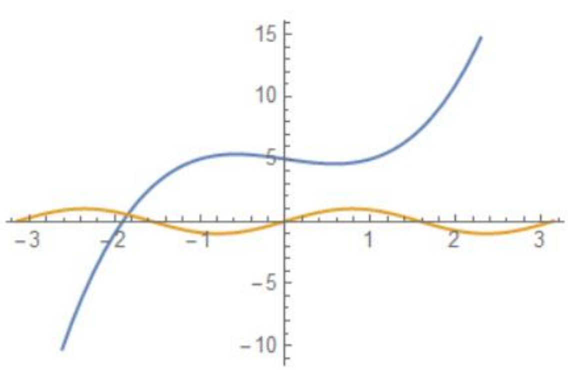


Figure 2: *Solution of Questions 2 using Wolfram Mathematica*

Answer: The root is located in $[-2,-1]$.

5.4 Small Practical Project

Solution

The root must be in the interval $[a,b]=[0,1]$, since the solution sought x is a percentage. The check of signs gives: $f(0)f(1)=(-2)(0.368...)<0$. If at the ends of the interval the continuous function $f(x)$ has different signs, then there is at least one real root in that interval. We compose a bisection method algorithm for Wolfram Mathematica in two steps:

a) Localization of the root.

```
Clear[k, x, f]
p = 1;  k = 1.648;
f[x_] := (p/k^2 - 1) * x^3 + 3 x - 2                (* Define the function *)
f[0]                        (* Calculate and print the values in 0 and 1 *)
f[1]
Plot[f[x], {x, -3, 3}]               (* Plot the function in [-3, 3] *)

-2.

0.368202
```

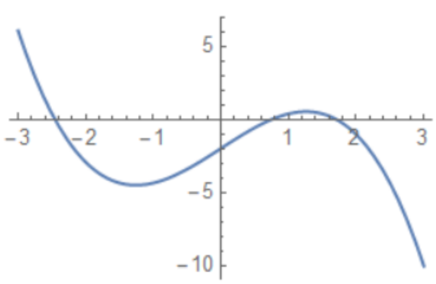


Figure 3: *Wolfram Mathematica Code for Localizing the Real Root to Solve the Project*

The plot clearly shows that the equation has three real roots, but only one is in the interval $[a,b]=[0,1]$.

b) Software implementation of the bisection method.

We compose a simple code in Wolfram Mathematica environment. The results after 10 iterations to reduce the length of the initial interval 10 times are shown in Figure 4.

```
a = 0; b = 1; eps = 0.001; i = 0; (* Set the interval and the accuracy ∈ *)
While[Abs[b - a] > eps,          (* Loop for cheching the current accuracy *)
  c = (a + b)/2.;        (* Calculate the middle point c of the current interval [a, b]*)
  If[ f[a] * f[c] < 0, b = c, a = c]; (* Check the signs of the function in
                    points a,c,b and choose the subinterval with different signs
                    of the function at the end points *)
  i = i + 1;                          (* Increment i for the next iteration *)
  Print[" i=", i, "  a=", a, "  b=", b, "  f[c]=", f[c]]
]
 i=1  a=0.5  b=1  f[c]=-0.578975
 i=2  a=0.75  b=1  f[c]=-0.01654
 i=3  a=0.75  b=0.875  f[c]=0.201744
 i=4  a=0.75  b=0.8125  f[c]=0.0986179
 i=5  a=0.75  b=0.78125  f[c]=0.042485
 i=6  a=0.75  b=0.765625  f[c]=0.0133268
 i=7  a=0.757813  b=0.765625  f[c]=-0.00151892
 i=8  a=0.757813  b=0.761719  f[c]=0.00592597
 i=9  a=0.757813  b=0.759766  f[c]=0.00220902
 i=10  a=0.757813  b=0.758789  f[c]=0.00034642
```

Figure 4: *Iteration Procedure for Solving the Project*

Answer: The approximate value of the root is $x_{10} \approx \frac{a+b}{2} \approx 0.758$ or $x \approx 75.8\%$ found with absolute accuracy 0.001. To note, this accuracy is satisfactory, since the initial datum $(k = 1.648)$ is given with accuracy of three decimal places.

Remark. All provided programming codes, tables and results are our own authors work.

5. Conclusion

Assessment of mathematical competences in higher education implies a complex approach involving various forms - exam, project, use of modern software, situational games, requiring mathematical solution, etc. This paper develops an approach for a competency assessment model, an outcome from a European project involving the experience of partner universities and research organizations from eight European countries. A specific detailed example for solving nonlinear equations is provided. It can be used by all professors and tutors in higher education institutions to update the learning and assessment of students' knowledge and skills in mathematics, based on a competency-oriented approach.

The proposed model can be used for many other mathematical subjects. It can be combined with various methods of teaching mathematics, both traditional and ICT enhanced, game-oriented, project-oriented and others. As a future work in the presented field, it is planned to develop a guide with a large number of pedagogical materials for assessment of mathematical competencies in engineering degrees.

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