Conference Name: Pure & Applied Sciences International Conference, 14-15 March 2024, Singapore
Conference Dates: 14-Mar- 2024 to 15-Mar- 2024
Conference Venue: The National University of Singapore Society (NUSS), Suntec City Guild House, Singapore
Appears in: MATTER: International Journal of Science and Technology (ISSN 2454-5880)
Publication year: 2024

Yanpei Zhen, 2024

Volume 2024, pp. 20-34

DOI- https://doi.org/10.20319/icstr.2024.2034

This paper can be cited as: Zhen, Y. (2024). Rogue Waves Arising on the Standing Periodic Wave in the High-Order Ablowitz-Ladik Equation. Pure & Applied Sciences International Conference, 14-15 March 2024, Singapore. Proceedings of Scientific and Technical Research Association (STRA), 2024, 20-34.

ROGUE WAVES ARISING ON THE STANDING PERIODIC WAVE IN THE HIGH-ORDER ABLOWITZ-LADIK EQUATION

Yanpei Zhen
School of Mathematics, Southeast University, Nanjing, China
230218752@seu.edu.cn

Abstract

The nonlinear Schrödinger (NLS) equation models wave dynamics in many physical problems related to fluids, plasmas, and optics. The standing periodic waves are known to be modulationally unstable and rogue waves (localized perturbations in space and time) have been observed on their backgrounds in numerical experiments. The exact solutions for rogue waves arising on the periodic standing waves have been obtained analytically. It is natural to ask if the rogue waves persist on the standing periodic waves in the integrable discretizations of the integrable NLS equation. We study the standing periodic waves in the semidiscrete integrable system modeled by the high-order Ablowitz-Ladik (AL) equation. The standing periodic wave of the high-order AL equation is expressed by the Jacobi cnoidal elliptic function. The exact solutions are obtained by using the separation of variables and one-fold Darboux transformation. Since the cnoidal wave is modulationally unstable, the rogue waves generated on the periodic background.

Keywords

Standing Periodic Waves, Rogue Waves, Darboux Transformation, High-Order

Introduction

Rogue waves have gained more and more attention recently [1]. In order to construct rogue waves on the periodic background, Chen and Pelinovsky first combine the nonlinearization of spectral problem with the Darboux transformation method [2], and then by using these two approaches, rogue waves on the periodic background have been obtained for the NLS equation [3, 4], mKdV equation [2, 5], derivative NLS equation [6, 7], sine-Gordon equation [8] and discrete mKdV equation [9, 10].

Two families of periodic wave solutions of NLS equation are constructed by the Jacobi elliptic functions [3] and modulational stability of these solutions with respect to long pertur- bations was studied in [11], where it was concluded that the dnoidal and cnoidal waves are modulationally unstable, rogue waves generated on the periodic background. Recently, they generalized this results to the Ablowitz-Ladik equation and investigated modulational stability of the standing periodic waves and obtained similar results [12]

In this paper, we consider the high-order AL equation in the following form

$$\dot{u}_n = i(1+|u_n|^2)[(1+|u_{n+1}|^2)u_{n+2} + (1+|u_{n-1}|^2)u_{n-2} + \bar{u}_n(u_{n-1}^2 + u_{n+1}^2) + u_n(\bar{u}_{n-1}u_{n+1} + u_{n-1}\bar{u}_{n+1})], \quad n \in \mathbb{Z}, \tag{1.1}$$

we construct new solutions on the periodic background of the equation (1.1) by combining the separation of variables and the Darboux transformation method. First, by using the separa- tion of variables, we obtain the forth-order difference equation and then we specify the exact expressions between the squared eigenfunctions and the standing periodic wave solution which is expressed by cnoidal elliptic function. Second, the cnoidal standing periodic wave can be obtained from the fourth-order difference equation. Finally, one-fold Darboux transformation can be used to construct the rogue waves generated on the cnoidal wave background.

The article is organized as follows. In section 2, we give details of the periodic squared eigenfunctions of high-order AL equation spectral problem related to the cnoidal elliptic func- tion. In section 3, we compute the standing periodic wave given by cnoidal elliptic function. In section 4, we compute the second, linearly independent solution of the Lax equations. We con- struct the rogue waves generated on the cnoidal wave background using the one-fold Darboux transformations in sections 5. Section 6 gives the conclusion.

The Separation of Variables

The equation (1.1) can be represented as the compatibility condition for the following Lax pair of linear equations

$$\psi_{n+1} = U_n\psi_n, \quad U_n = \frac{1}{\sqrt{1+|u_n|^2}}\begin{pmatrix}\lambda & u_n\\ -\bar{u}_n & \lambda^{-1}\end{pmatrix}, \tag{2.1}$$

and

$$i\dot{\psi}_n = V_n\psi_n, \quad V_n = \begin{pmatrix}V_n^{11} & V_n^{12}\\ V_n^{21} & -V_n^{11}\end{pmatrix}, \tag{2.2}$$

where

$$V_n^{11} = \frac{1}{2}(\lambda^4+\lambda^{-4}) + \bar{u}_{n-1}u_n\lambda^2 + u_{n-1}\bar{u}_n\lambda^{-2} + \frac{1}{2}[\bar{u}_{n-1}^2u_n^2 + u_{n-1}^2\bar{u}_n^2$$
$$+(1+|u_{n-1}|^2)(\bar{u}_{n-2}u_n + u_{n-2}\bar{u}_n) + (1+|u_n|^2)(\bar{u}_{n-1}u_{n+1} + u_{n-1}\bar{u}_{n+1})],$$
$$V_n^{12} = \lambda^3u_n + \lambda[\bar{u}_{n-1}u_n^2 + (1+|u_n|^2)u_{n+1}] - \lambda^{-3}u_{n-1} - \lambda^{-1}[u_{n-1}^2\bar{u}_n + (1+|u_{n-1}|^2)u_{n-2}],$$
$$V_n^{21} = -\lambda^3\bar{u}_{n-1} - \lambda[\bar{u}_{n-1}^2u_n + (1+|u_{n-1}|^2)\bar{u}_{n-2}] + \lambda^{-3}\bar{u}_n + \lambda^{-1}[u_{n-1}\bar{u}_n^2 + (1+|u_n|^2)\bar{u}_{n+1}].$$

We consider the standing wave solution of the equation (1.1) in the form

$$u_n = U_ne^{2i\omega t}, \tag{2.3}$$

where U_n is the real periodic function and ω is a real parameter.

Substituting (2.3) into the high-order AL equation (1.1), we obtain the forth-order difference equation

$$(1 + U_n^2)[(1 + U_{n+1}^2)U_{n+2} + (1 + U_{n-1}^2)U_{n-2} + U_n(U_{n-1}^2 + U_{n+1}^2) + 2U_{n-1}U_nU_{n+1}] = 2\omega U_n, \quad n \in Z. \tag{2.4}$$

Let us separate the variables for solutions $\psi_n = (p_n, q_n)^T$ of the Lax equations (2.1) and (2.2)

$$p_n = P_n(t)e^{i\omega t}, \; q_n = Q_n(t)e^{-i\omega t}. \tag{2.5}$$

Substituting (2.3) and (2.5) into the Lax equations (2.1) and (2.2), we obtain the following Lax equations

$$\begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix} = \frac{1}{\sqrt{1+U_n^2}} \begin{pmatrix} \lambda & U_n \\ -U_n & \lambda^{-1} \end{pmatrix} \begin{pmatrix} P_n \\ Q_n \end{pmatrix}, \tag{2.6}$$

and

$$\frac{d}{dt} \begin{pmatrix} P_n \\ Q_n \end{pmatrix} = i \begin{pmatrix} \tilde{V}^{11}_n & \tilde{V}^{12}_n \\ \tilde{V}^{21}_n & -\tilde{V}^{11}_n \end{pmatrix} \begin{pmatrix} P_n \\ Q_n \end{pmatrix}, \tag{2.7}$$

where

$$\begin{aligned}
\tilde{V}^{11}_n =& \frac{1}{2}(\lambda^4 + \lambda^{-4}) + U_{n-1}U_n(\lambda^2+\lambda^{-2}) + U_{n-1}U_n^2 + U_{n-2}U_n(1+U_{n-1}^2) \\
&+ U_{n-1}U_{n+1}(1+U_n^2) - \omega, \\
\tilde{V}^{12}_n =& \lambda^3 U_n + \lambda[U_{n-1}U_n^2 + (1+U_n^2)U_{n+1}] - \lambda^{-3}U_{n-1} \\
&- \lambda^{-1}[U_{n-1}^2 U_n + (1+U_{n-1}^2)U_{n-2}], \\
\tilde{V}^{21}_n =& -\lambda^3 U_{n-1} - \lambda[U_{n-1}^2 U_n + (1+U_{n-1}^2)U_{n-2}] + \lambda^{-3}U_n \\
&+ \lambda^{-1}[U_{n-1}U_n^2 + (1+U_n^2)U_{n+1}].
\end{aligned}$$

Lemma 1 *Let U_n be a solution of the forth-order difference equation (2.4). The real-valued quantities*

$$\begin{aligned}
F_1 =& 2(U_{n-2}U_n + U_{n-1}^2U_n^2 + U_{n-1}U_{n+1} + U_{n-2}U_{n-1}^2U_n + U_{n-1}U_n^2U_{n+1})\omega \\
&- U_{n-2}^2 - U_{n-1}^2 - U_n^2 - U_{n+1}^2 - 2U_n^2U_{n+1}^2 - U_n^4U_{n+1}^2 - U_n^2U_{n+1}^4 - U_{n-1}^4U_n^2 - U_{n-1}^2U_n^4 \\
&- U_{n-1}^4U_n^4 - 2U_{n-1}^2U_n^2 - U_{n-2}^2U_n^2 - U_{n-1}^2U_{n+1}^2 - U_{n-2}^2U_{n-1}^4 - 2U_{n-2}^2U_{n-1}^2 \\
&- 2U_{n-1}U_nU_{n+1} - 2U_{n-1}U_n^4U_{n+1} - 2U_{n-1}^3U_n^2U_{n+1} - 2U_{n-1}^2U_n^2U_{n+1}^3 \\
&- 2U_{n-1}U_n^2U_{n+1} - 2U_{n-2}U_{n-1}^4U_n^2 - 2U_{n-1}^2U_nU_{n+1} - 2U_{n-2}U_{n-1}^2U_n \\
&- 2U_{n-2}U_{n-1}U_n - 2U_{n-2}U_{n-1}^2U_n - 2U_{n-2}U_{n-1}^3U_n - 2U_{n-2}U_{n-1}U_n^3 \\
&- 2U_{n-2}U_{n-1}U_nU_{n+1} - 2U_{n-2}U_{n-1}^3U_nU_{n+1} - 2U_{n-2}U_{n-1}U_n^3U_{n+1} \\
&- 2U_{n-2}U_{n-1}U_nU_{n+1},
\end{aligned}$$

and

$$\begin{aligned}
F_2 =& 2\omega U_{n-1}U_n - U_n^3U_{n+1} - U_nU_{n+1} - U_{n-1}U_n - U_{n-1}^3U_n^3 - U_{n-1}U_n^3 \\
&- U_{n-1}^3U_n + U_{n-2}U_{n+1} - U_{n-2}U_{n-1} - U_{n-2}U_{n-1}^3 - U_{n-1}U_n^2U_{n+1} \\
&- U_{n-1}U_nU_{n+1} + U_{n-2}U_nU_{n+1} - U_{n-2}U_{n-1}U_n - U_{n-2}U_{n-1}^2U_n \\
&+ U_{n-2}U_{n-1}^2U_{n+1} + U_{n-2}U_{n-1}^2 + U_n^2U_{n+1}.
\end{aligned}$$

are independent of $n \in \mathbb{Z}$.

Proof : It is easy to verify that $(E-1)F_i = 0$, $i = 1, 2$, where we use the following shift operators

$$E(\psi_n) = \psi_{n+1},\; E^{-1}(\psi_n) = \psi_{n-1}.$$

Proposition 2 *If the Lax equations (2.6) and (2.7) are solved with the separation of variables as*

$$P_n(t) = \tilde{P}_n e^{\Omega t}, \qquad Q_n(t) = \tilde{Q}_n e^{\Omega t}, \tag{2.8}$$

where $(\tilde{P}_n, \tilde{Q}_n)^T$ *is t-independent, then the spectral parameters* Ω *and* λ *are related by the algebraic equation*

$$\Omega^2 + P(\lambda) = 0, \tag{2.9}$$

where

$$P(\lambda) = \frac{1}{4}(\lambda^{8} + \lambda^{-8}) - \omega(\lambda^{4} + \lambda^{-4}) - F_2(\lambda^{2} + \lambda^{-2}) - F_1 + \omega^{2} + \frac{1}{2}. \quad (2.10)$$

Proof: Substituting (2.8) into the time-evolution problem (2.7), we obtain a linear algebraicsystem

$$i\begin{pmatrix} \tilde{V}^{11}_n + i\Omega & \tilde{V}^{12}_n \\ \tilde{V}^{21}_n & -\tilde{V}^{11}_n + i\Omega \end{pmatrix}\begin{pmatrix} \tilde{P}_n \\ \tilde{Q}_n \end{pmatrix} = 0,$$

which admits a nonzero solution if and only if the determinant of the coefficient matrix is zero

$$\begin{vmatrix} \tilde{V}^{11}_n + i\Omega & \tilde{V}^{12}_n \\ \tilde{V}^{21}_n & -\tilde{V}^{11}_n + i\Omega \end{vmatrix} = 0.$$

Expanding the determinant yields the algebraic equation in the form (2.9) and (2.10), whichcompletes the proof.

Proposition 3 *Let $\lambda_1 \in \mathbb{C}$ be a root of the polynomial $P(\lambda)$ in (2.10) and define*

$$\omega = \frac{1}{2}(\lambda_1^{4} + \lambda_1^{-4}) + \sigma_1\sqrt{F_1 + (\lambda_1^{2} + \lambda_1^{-2})F_2}, \quad \sigma_1 = \pm 1. \quad (2.11)$$

Then, the eigenfunction $(P_n, Q_n)^T$ of the Lax equations (2.6) and (2.7) with λ_1 is given by

$$\begin{aligned} P_n^2 = {} & \lambda_1^3 U_n + \lambda_1[U_{n-1}U_n^2 + (1 + U_n^2)U_{n+1}] - \lambda_1^{-3}U_{n-1} \\ & - \lambda_1^{-1}[U_{n-1}^2 U_n + (1 + U_{n-1}^2)U_{n-2}], \end{aligned} \quad (2.12)$$

$$\begin{aligned} Q_n^2 = {} & \lambda_1^3 U_{n-1} + \lambda_1[U_{n-1}^2 U_n + (1 + U_{n-1}^2)U_{n-2}] - \lambda_1^{-3}U_n \\ & - \lambda_1^{-1}[U_{n-1}U_n^2 + (1 + U_n^2)U_{n+1}], \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} P_nQ_n = {} & \sigma_1\sqrt{F_1 + (\lambda_1^2 + \lambda_1^{-2})F_2} - U_{n-1}U_n(\lambda_1^2 + \lambda_1^{-2}) - U_{n-1}^2 U_n^2 \\ & - U_{n-2}U_n(1 + U_{n-1}^2) - U_{n-1}U_{n+1}(1 + U_n^2). \end{aligned} \quad (2.14)$$

Proof: The relation (2.11) is given by solving $P(\lambda_1) = 0$ in ω. Since the root of $P(\lambda)$corresponds to $\Omega = 0$, it follows from (2.7) that P_n and Q_n are related by

$$\begin{aligned} & [\tfrac{1}{2}(\lambda_1^4 + \lambda_1^{-4}) + U_{n-1}U_n(\lambda_1^2 + \lambda_1^{-2}) + U_{n-1}^2U_n^2 + U_{n-2}U_n(1 + U_{n-1}^2) \\ & + U_{n-1}U_{n+1}(1 + U_n^2) - \omega]P_n + [\lambda_1^3 U_n + \lambda_1[U_{n-1}U_n^2 + (1 + U_n^2)U_{n+1}] \\ & - \lambda_1^{-3}U_{n-1} - \lambda_1^{-1}[U_{n-1}^2 U_n + (1 + U_{n-1}^2)U_{n-2}]]Q_n = 0. \end{aligned}$$

Multiplying this relation by P_n and by Q_n verifies the relations (2.12)-(2.14) with the help ofrelations (2.11), which completes the proof .

1. Cnoidal Standing Periodic Wave

There exists the exact standing periodic wave solution of the fourth-order difference equa-tion (2.4) in the form of the Jacobi cnoidal elliptic function

$$U_n(t) = A cn(\alpha n, k), \tag{3.1}$$

where $\alpha \in (0, 2K(k)), k \in (0, 1)$ are arbitrary parameters. Substituting (3.1) into (2.4), we obtain

$$A = \frac{k\,sn(\alpha, k)}{dn(\alpha, k)}, \quad \omega = \frac{1 + 2(k^2 - 1)sn^2(\alpha, k) - k^2 sn^4(\alpha, k)}{dn^4(\alpha, k)}. \tag{3.2}$$

Considering the conserved quantity F_1 and F_2 at $\alpha n = 0$ yields

$$\begin{aligned} F_1 &= \frac{1}{dn^8(\alpha, k)}\Big((-8k^2 + 8k^4)sn^4(\alpha, k) + (8k^2 - 8k^4)sn^6(\alpha, k) \\ &\quad + (k^4 - 2k^6 + k^8)sn^8(\alpha, k) \Big), \\ F_2 &= \frac{1}{dn^6(\alpha, k)} 4k^2(k^2 - 1)sn^4(\alpha, k)cn(\alpha, k). \end{aligned} \tag{3.3}$$

Substituting (3.2) and (3.3) into (2.11), we obtain

$$\lambda_1 = \frac{\sqrt{(1 - k\,sn(\alpha, k))(cn(\alpha, k) + i\sqrt{1 - k^2 sn^2(\alpha, k)})}}{dn(\alpha, k)}, \quad \sigma_1 = +1. \tag{3.4}$$

2. Nonperiodic solution of the Lax equations

The following lemma presents the nonperiodic solution of the Lax equations (2.6) and (2.7) for the eigenvalue λ_1.

Lemma 4 *Let $(P_n, Q_n)^T$ be a solution to the Lax equations (2.6) and (2.7) for the eigenvalue $\lambda = \lambda_1$ given by roots of the polynomial $P(\lambda)$. The second, linearly independent solution $(\hat{P}_n, \hat{Q}_n)$ of the Lax equations (2.6) and (2.7) with the same eigenvalue $\lambda = \lambda_1$ is denoted by*

$$\hat{P}_n = P_n\theta_n - \frac{\bar{Q}_n}{|P_n|^2 + |Q_n|^2}, \quad \hat{Q}_n = Q_n\theta_n + \frac{\bar{P}_n}{|P_n|^2 + |Q_n|^2}, \tag{4.1}$$

where

$$\theta_{n+1} - \theta_n = \frac{(|\lambda_1|^2 - 1)(\bar{\lambda}_1 U_n \bar{P}_n^2 - \lambda_1 U_n \bar{Q}_n^2 - (1 + |\lambda_1|^2)\bar{P}_n\bar{Q}_n)}{Y_n}, \tag{4.2}$$

where

$$\begin{aligned} Y_n &= |\lambda_1|^4 P_n^2\bar{P}_n^2 + |\lambda_1|^2 U_n^2 P_n^2 \bar{P}_n^2 + |\lambda_1|^2 \lambda_1 U_n P_n^2 \bar{P}_n \bar{Q}_n - \lambda_1 U_n P_n^2 \bar{P}_n \bar{Q}_n \\ &+ |\lambda_1|^2 \bar{\lambda}_1 U_n \bar{P}_n^2 P_n Q_n - \bar{\lambda}_1 U_n \bar{P}_n^2 P_n Q_n + |\lambda_1|^4 \bar{P}_n \bar{Q}_n P_n Q_n + 2|\lambda_1|^2 U_n^2 \bar{P}_n \bar{Q}_n P_n Q_n \\ &+ \bar{P}_n \bar{Q}_n P_n Q_n + |\lambda_1|^2 \bar{\lambda}_1 U_n \bar{P}_n \bar{Q}_n Q_n^2 - \bar{\lambda}_1 U_n \bar{P}_n \bar{Q}_n Q_n^2 + |\lambda_1|^2 \lambda_1 U_n P_n Q_n \bar{Q}_n^2 \\ &- \lambda_1 U_n P_n Q_n \bar{Q}_n^2 + |\lambda_1|^2 U_n^2 Q_n^2 \bar{Q}_n^2 + Q_n^2 \bar{Q}_n^2, \end{aligned}$$

and

$$\theta_{n,t} = i. \tag{4.3}$$

Proof: Substituting (4.1) into (2.6) and (2.7), after long but straightforward computations,we have simplified the expression to the form (4.2) and (4.3).

Remark 1 *It follows from (4.2) and (4.3) that* $\theta_n(t) = \Theta_n + it$*, where the t-independent* $\Theta_n(t)$
is obtained from the difference equation (4.2).

3. Rogue Waves on the Cnoidal Wave Background

The following lemma presents the one-fold Darboux transformation for the high-order AL equation (1.1).

Lemma 5 *Let $\psi_n = (p_n, q_n)^T$ for the eigenvalue λ_1 be a solution to the Lax equations (2.1) and (2.2) pertinent to the spectral $u_n(t)$ of the equation (1.1), then*

$$\hat{u}_n = \frac{\lambda_1(1-|\lambda_1|^4)p_n\bar{q}_n - (|\lambda_1|^2|p_n|^2 + |\lambda_1|^4|q_n|^2)u_n}{\bar{\lambda}_1^2(|\lambda_1|^2|p_n|^2 + |q_n|^2)}, \tag{5.1}$$

is a new solution of the equation (1.1) and $\hat{\psi}_n = T_n^{[1]}\psi_n$ is a new solution to the Lax equations (2.1) and (2.2) with arbitrary λ, where

$$T_n^{[1]} = \frac{1}{\sqrt{-a_n}}\begin{pmatrix} \lambda + a_n\lambda^{-1} & b_n \\ -\bar{b}_n & \lambda^{-1} + \bar{a}_n\lambda \end{pmatrix},$$

with

$$a_n = -\frac{\lambda_1(|\lambda_1|^2|p_n|^2 + |q_n|^2)}{\bar{\lambda}_1(|p_n|^2 + |\lambda_1|^2|q_n|^2)}, \quad b_n = \frac{\lambda_1(1-|\lambda_1|^4)p_n\bar{q}_n}{|\lambda_1|^2(|p_n|^2 + |\lambda_1|^2|q_n|^2)}.$$

Proof: By using Wolframs Mathematica symbolic computations, it is easy to verify that $T_n^{[1]}$ satisfies Darboux equations $T_{n+1}^{[1]}U_n = \hat{U}_n T_n^{[1]}$ and $\hat{V}_n T_n^{[1]} = T_{n,t}^{[1]} + T_n^{[1]}V_n$.

Remark 2 *For cnoidal wave (3.1), substituting (2.3) and (2.5) into the one-fold Darboux transformation (5.1), we get*

$$\hat{u}_n = \hat{U}_n e^{2i\omega t}, \tag{5.2}$$

with

$$\hat{U}_n = -\frac{\lambda_1(|P_n|^2 + |\lambda_1|^2|Q_n|^2)U_n}{\bar{\lambda}_1(|\lambda_1|^2|P_n|^2 + |Q_n|^2)} + \frac{\lambda_1(1-|\lambda_1|^4)P_n\bar{Q}_n}{\bar{\lambda}_1^2(|\lambda_1|^2|P_n|^2 + |Q_n|^2)}. \tag{5.3}$$

Substituting (5.1) into (5.3), we get the rogue wave solution to the equation (1.1) in the analytic form

$$\hat{u}_n = -\frac{\lambda_1(|\hat{P}_n|^2 + |\lambda_1|^2|\hat{Q}_n|^2)U_n}{\bar{\lambda}_1(|\lambda_1|^2|\hat{P}_n|^2 + |\hat{Q}_n|^2)} + \frac{\lambda_1(1-|\lambda_1|^4)\hat{P}_n\bar{\hat{Q}}_n}{\bar{\lambda}_1^2(|\lambda_1|^2|\hat{P}_n|^2 + |\hat{Q}_n|^2)} - \frac{\Theta_1}{\Theta_2}, \tag{5.4}$$

where

$$\Theta_2 = \bar{\lambda}_1^2\,[|\lambda_1|^2\bar{P}_n^2(P_n^2\bar{P}_n^2 P_nQ_n + 2P_n^2Q_n^2\bar{P}_n\bar{Q}_n + Q_n^2\bar{Q}_n^2P_nQ_n) + Q_n^2(P_n^2\bar{P}_n^2\bar{P}_n\bar{Q}_n + 2\bar{P}_n^2\bar{Q}_n^2P_nQ_n + Q_n^2\bar{Q}_n^2\bar{P}_n\bar{Q}_n)]|\theta_n|^2 + (1-|\lambda_1|^2)[Q_n^2(P_n^2\bar{P}_n^2 + P_nQ_n\bar{P}_n\bar{Q}_n)\theta_n + \bar{P}_n^2(P_nQ_n\bar{P}_n\bar{Q}_n + Q_n^2\bar{Q}_n^2)\bar{\theta}_n] + |\lambda_1|^2Q_n^2\bar{P}_n\bar{Q}_n + \bar{P}_n^2P_nQ_n,$$

$$
\begin{aligned}
\Theta_1 = {} & \lambda_1(1 - |\lambda_1|^4)[(P_n^2 \bar{P}_n^2 P_n Q_n \bar{P}_n \bar{Q}_n + 2P_n^2 \bar{P}_n^2 Q_n^2 \bar{Q}_n^2 + Q_n^2 \bar{Q}_n^2 P_n Q_n \bar{P}_n \bar{Q}_n)|\theta_n|^2 \\
& + P_n^2(\bar{P}_n^2 P_n Q_n + Q_n^2 \bar{P}_n \bar{Q}_n)\theta_n - \bar{Q}_n^2 (\bar{P}_n^2 P_n Q_n + Q_n^2 \bar{P}_n \bar{Q}_n)\bar{\theta}_n - P_n Q_n \bar{P}_n \bar{Q}_n] \\
& - |\lambda_1|^2 [\bar{P}_n^2(P_n^2 \bar{P}_n^2 P_n Q_n + 2P_n^2 Q_n^2 \bar{P}_n \bar{Q}_n + Q_n^2 \bar{Q}_n^2 P_n Q_n) + |\lambda_1|^2 Q_n^2 (P_n^2 \bar{P}_n^2 \bar{P}_n \bar{Q}_n \\
& + 2\bar{P}_n^2 \bar{Q}_n^2 P_n Q_n + Q_n^2 \bar{Q}_n^2 \bar{P}_n \bar{Q}_n)]|\theta_n|^2 + (|\lambda_1|^2 - 1)[Q_n^2 (P_n^2 \bar{P}_n^2 + P_n Q_n \bar{P}_n \bar{Q}_n)\theta_n \\
& + \bar{P}_n^2(P_n Q_n \bar{P}_n \bar{Q}_n + Q_n^2 \bar{Q}_n^2)\bar{\theta}_n] + Q_n^2 \bar{P}_n \bar{Q}_n + |\lambda_1|^2 \bar{P}_n^2 P_n Q_n \\
& U_n.
\end{aligned}
$$

and P_n^2, Q_n^2 and P_nQ_n are given by (2.12),(2.13) and (2.14).
Figure 1 shows that when we choose $\alpha = \frac{K(k)}{4}$ and $k = 0.999$, the solution surface of the
rogue wave solutions (5.4) arising on the cnoidal wave (3.1) for the eigenvalue λ_1 given by (3.4).

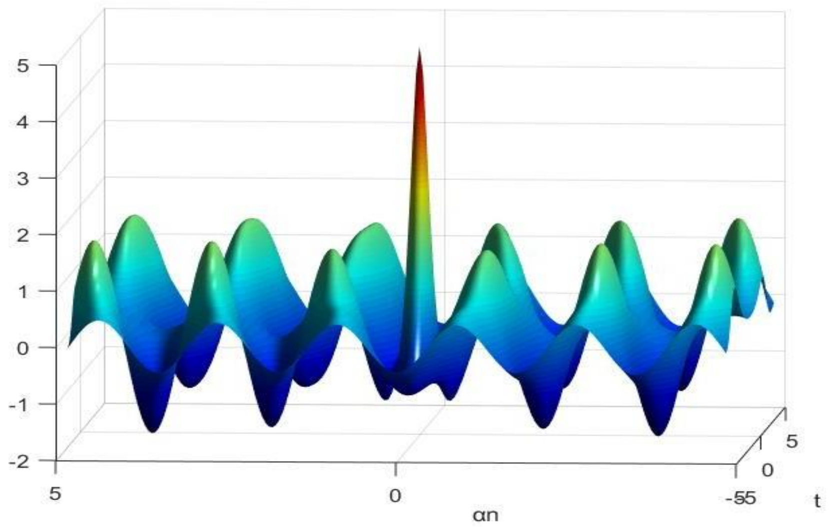

Figure 1: The solution surface for the rogue wave solutions arising on the background of thecnoidal wave with $\alpha = K(k)/4$ and $k = 0.999$.

4. Conclusion

In this paper, we construct the exact solutions for the high-order AL equation. Since the cnoidal periodic wave is modulationally unstable, we use the one-fold Darboux transformations to construct the rogue wave solutions arising on cnoidal wave background.

REFERENCES

A. V. Slunyaev, D. E. Pelinovsky, E. N. Pelinovsky, Rogue waves in the sea: observations,physics and mathematics, Phys. Usp 66 (2023) 148–172.

J. B. Chen, D. E. Pelinovsky, Rogue periodic waves of the modified KdV equation, Non- linearity 31 (2018) 1955–1980.

J. B. Chen, D. E. Pelinovsky, Rogue periodic waves in the focusing nonlinear Schrödinger equation, Proc. R. Soc. A 474 (2018) 20170814.

J. B. Chen, D. E. Pelinovsky, R. E. White, Periodic standing waves in the focusing non- linear Schrödinger equation: Rogue waves and modulation instability, Phys. D 405 (2020) 132378.

J. B. Chen, D. E. Pelinovsky, Periodic travelling waves of the Modified KdV equation androgue waves on the periodic background, J. Nonlinear Sci. 29 019)

2797–2843.

J. B. Chen, D. E. Pelinovsky, J. Upsal, Modulational instability of periodic standing waves in the derivative NLS equation, J. Nonlinear Sci. 31 (2021) 58

J. B. Chen, D. E. Pelinovsky, Rogue waves on the background of periodic standing waves in the derivative nonlinear Schrödinger equation, Phys. Rev. E 103 (2021) 062206.

D. E. Pelinovsky, R. E. White, Localized structures on librational and rotational travelling waves in the sine-Gordon equation, Proc. R. Soc. A 476 (2020) 20200490.

J. B. Chen, D. E. Pelinovsky, Periodic waves in the discrete mKdV equation: Modulational instability and rogue waves, Phys. D 445 (2023) 133652.

Y. P. Zhen, J. B. Chen, Rogue waves on the periodic background in the high-order discrete mKdV equation, Nonlinear Dyn. 111 (2023) 12511–12524.

B. Deconinck, B. L. Segal, The stability spectrum for elliptic solutions to the focusing NLS equation, Phys. D 346 (2017) 1–19.

J. B. Chen, D. E. Pelinovsky, Rogue waves arising on the standing periodic waves in the Ablowitz-Ladik equation, Stud. Appl. Math. in press.